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The mean value of the product of class numbers of paired quadratic fields III

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Abstract

This is the final part in a series of three papers. In this part, we evaluate the previously unevaluated local densities at dyadic places that appear in the density theorem stated in the first part. For this purpose we introduce an invariant, the level, attached to a pair of ramified quadratic extensions of a dyadic local field. This invariant measures how close the fields are in their arithmetic properties and its evaluation may be of interest independent of its application here.

Keywords: Density theorem; Prehomogeneous vector space; Binary hermitian forms; Local zeta functions

1. Introduction

This is the final part in a series of three papers. We begin by recalling the main result of the series, whose proof will be completed here. If k is a number field, let Δ_k, h_k, and R_k be the absolute discriminant (which is an integer), the class number, and the regulator, respectively. We fix a number field k and a quadratic extension $\tilde{k}$ of k. If $F\neq\tilde{k}$ is another quadratic extension of k, let $\tilde{F}$ be the compositum of F and $\tilde{k}$. Then $\tilde{F}$ is a biquadratic extension of k and so contains precisely three quadratic extensions, $\tilde{k}$, F and, say, F^* of k. We say that F and F^* are *paired*.

For simplicity we specialize to the case $k=\mathbb{Q}$. Let $\tilde{k}=\mathbb{Q}(\sqrt{d_0})$ where $d_0\neq 1$ is a square free integer. Suppose $|\Delta_{\mathbb{Q}(\sqrt{d_0})}|=\prod_p p^{\tilde{\delta}_p(d_0)}$ is the prime decomposition. For

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any prime number p, we put

$$E'_p(d_0) = \begin{cases} 1 - 3p^{-3} + 2p^{-4} + p^{-5} - p^{-6}, \\ (1+p^{-2})(1-p^{-2}-p^{-3}+p^{-4}), \\ (1-p^{-1})(1+p^{-2}-p^{-3}+p^{-2\tilde{\delta}_p(d_0)-2\lfloor \tilde{\delta}_p(d_0)/2 \rfloor -1}), \end{cases}$$

when p is, respectively, split, inert or ramified in $\tilde{k}$, where $\lfloor \tilde{\delta}_p(d_0)/2 \rfloor$ is the largest integer less than or equal to $\tilde{\delta}_p(d_0)/2$. We define

$$c_+(d_0) = \begin{cases} 16 & d_0 > 0, \\ 8\pi & d_0 < 0, \end{cases} \quad c_-(d_0) = \begin{cases} 4\pi^2 & d_0 > 0, \\ 8\pi & d_0 < 0, \end{cases}$$

$$M(d_0) = |\Delta_{\mathbb{Q}(\sqrt{d_0})}|^{\frac{1}{2}} \zeta_{\mathbb{Q}(\sqrt{d_0})}(2) \prod_p E'_p(d_0).$$

The following two theorems are our main results.

Theorem 1.1. *With either choice of sign we have*

$$\lim_{X \to \infty} X^{-2} \sum_{\substack{[F:\mathbb{Q}]=2, \\ 0 < \pm \Delta_F < X}} h_F R_F h_{F^*} R_{F^*} = c_\pm(d_0)^{-1} M(d_0).$$

Theorem 1.2. *With either choice of sign we have*

$$\lim_{X \to \infty} X^{-2} \sum_{\substack{[F:\mathbb{Q}]=2, \\ 0 < \pm \Delta_F < X}} h_{F(\sqrt{d_0})} R_{F(\sqrt{d_0})} = c_\pm(d_0)^{-1} h_{\mathbb{Q}(\sqrt{d_0})} R_{\mathbb{Q}(\sqrt{d_0})} M(d_0).$$

For a general introduction to this problem, the reader should see the introduction to Part I. Our method of deriving density theorems such as Theorems 1.1 and 1.2 from information on the zeta functions of prehomogeneous vector spaces is called the filtering process. The filtering process for this case was discussed in the introduction and Sections 6 and 7 of Part I. The remaining task for us to finish the filtering process is to find the previously unevaluated local densities at the dyadic places of k and this is the main purpose of this part.

Let W be the space of binary Hermitian forms. Our approach to the above theorems is based on a consideration of the zeta function for the following prehomogeneous vector space:

$$G = \mathrm{GL}(2)_{\tilde{k}} \times \mathrm{GL}(2), \quad V = W \otimes \mathrm{Aff}^2, \tag{1}$$

where $\mathrm{GL}(2)_{\tilde{k}}$ is regarded as a group over k by restriction of scalars and Aff^2 is affine 2-space regarded as a variety over k. There is a relative invariant polynomial $P(x)$ of degree four (defined immediately after (3.4) in Part I) and we put $V^{\mathrm{ss}} = \{x \in V \mid P(x) \neq 0\}$.

Let v be a finite place of k, k_v be the completion of k at this place and $K_v \subseteq G_{k_v}$ be the standard maximal compact subgroup of G_{k_v}. We assume that $\tilde{k}_v = \tilde{k} \otimes_k k_v$ is a field. It is proved in [1, p. 324] that the orbit space $G_{k_v} \backslash V_{k_v}^{\mathrm{ss}}$ corresponds bijectively with the set of extensions of k_v of degree one or two. For $x \in V_{k_v}^{\mathrm{ss}}$ we denote the field corresponding to x by $k_v(x)$ and the identity component of the stabilizer of x by G_x°.

In Part I we selected standard representatives for the orbits in $G_{k_v} \backslash V_{k_v}^{\mathrm{ss}}$ and introduced an equivalence relation $\asymp$ on $V_{k_v}^{\mathrm{ss}}$ whose equivalence classes are unions of G_{k_v}-orbits. These definitions will be reviewed, respectively, in Section 2 and at the end of Section 3. On V_{k_v} we use the additive Haar measure under which $\mathrm{vol}(V_{\mathcal{O}_v}) = 1$ and on $G_{x\,k_v}^{\circ}$ the Haar measure described in [3], Definition 5.13. We shall not have to recall this latter definition here; all the information we require about it will be presented at the beginning of Section 4. If x is the standard representative for an orbit in $G_{k_v} \backslash V_{k_v}^{\mathrm{ss}}$ then we define

$$\varepsilon_v(x) = \mathrm{vol}(G_{x\,k_v}^{\circ} \cap K_v)\,\mathrm{vol}(K_v x)$$

and

$$\bar{\varepsilon}_v(x) = \sum_{y \asymp x} \varepsilon_v(y),$$

where the sum is over standard representatives for orbits in the equivalence class of x. The *local density* at v is then

$$E_v = \sum_x \varepsilon_v(x) = \sum_x \bar{\varepsilon}_v(x),$$

where the first sum is over all standard representatives for orbits in $G_{k_v} \backslash V_{k_v}^{\mathrm{ss}}$ and the second over a set containing one standard representative for an orbit in each class in $G_{k_v} \backslash V_{k_v}^{\mathrm{ss}} / \asymp$. The values of $\bar{\varepsilon}_v(x)$ calculated in this paper are summarized in Tables 1 and 2. The remaining notation used in these tables is defined in Section 2 and at the end of Section 3. The values of $\bar{\varepsilon}_v(x)$ in Tables 1 and 2 are verified in Propositions 4.2, 5.11, 5.14, Corollary 5.15 and [2], Proposition 3.3.

All the cases we have to deal with here involve pairs $(\tilde{k}_v, k_v(x))$ of ramified quadratic extensions of k_v. Since v is dyadic, they are both wildly ramified and this is the main difficulty of the situation. The definition of $\varepsilon_v(x)$ consists of two factors, $\mathrm{vol}(G_{x\,k_v}^{\circ} \cap K_v)$ and $\mathrm{vol}(K_v x)$. It is the second factor which requires grouping of orbits to compute. So, for us to be able to compute $\bar{\varepsilon}_v(x)$, the first factor has be the same for all x in the same group. This means that the grouping has to be coarse enough to compute the sum of the second factors, but fine enough so that the first factor stays constant in every group. When we defined the appropriate grouping in Section 7 of Part I, we used the relative discriminants of the extensions $k_v(x)/k_v$ and $\tilde{k}_v(x)/\tilde{k}_v$,

Table 1
$\bar{\varepsilon}_v(x) = \varepsilon_v(x)$ for types (rm rm)* and (rm rm ur)

Index	$\bar{\varepsilon}_v(x)$
(rm rm)*	$\frac{1}{2}q_v^{-2\tilde{\delta}_v-2\lfloor\tilde{\delta}_v/2\rfloor}(1-q_v^{-2})^2$
(rm rm ur)	$q_v^{-2\tilde{\delta}_v}(1-\frac{1}{2}q_v^{-2\lfloor\tilde{\delta}_v/2\rfloor})(1-q_v^{-1})^2(1-q_v^{-2})$

Table 2
$\bar{\varepsilon}_v(x)$ for grouped dyadic orbits of type (rm rm rm)

Conditions	$\bar{\varepsilon}_v(x)$
$\delta_{x,v} \neq \tilde{\delta}_v,\ \delta_{x,v} \leqslant 2m_v$	$q_v^{-(\delta_{x,v}/2+\lambda_{x,v})}(1-q_v^{-1})^2(1-q_v^{-2})^2$
$\delta_{x,v} \neq \tilde{\delta}_v,\ \delta_{x,v} = 2m_v+1$	$q_v^{-(m_v+\lambda_{x,v}+1)}(1-q_v^{-1})(1-q_v^{-2})^2$
$\delta_{x,v} = \tilde{\delta}_v \leqslant 2m_v,\ \lambda_{x,v} = \frac{1}{2}\tilde{\delta}_v$	$q_v^{-2\lambda_{x,v}}(1-q_v^{-1})(1-2q_v^{-1})(1-q_v^{-2})^2$
$\delta_{x,v} = \tilde{\delta}_v \leqslant 2m_v,\ \lambda_{x,v} > \frac{1}{2}\tilde{\delta}_v$	$q_v^{-2\lambda_{x,v}}(1-q_v^{-1})^2(1-q_v^{-2})^2$
$\delta_{x,v} = \tilde{\delta}_v = 2m_v+1$	$q_v^{-2\lambda_{x,v}}(1-q_v^{-1})^2(1-q_v^{-2})^2$

where $\tilde{k}_v(x)$ is the compositum of $\tilde{k}_v$ and $k_v(x)$. However, we would like to use congruence conditions on the vector space V to compute the sum of $\mathrm{vol}(K_v x)$ and it is not easy to relate the relative discriminant of $\tilde{k}_v(x)/\tilde{k}_v$ directly to congruence conditions on V.

To surmount this difficulty, we introduce, in Section 2, the notion of the *level* of a pair (k_1, k_2) of ramified quadratic extensions of k_v. This number provides a measure of how close k_1 and k_2 are in their arithmetic properties and we prove that the grouping with respect to the level is the same as the grouping with respect to the relative discriminants of $k_v(x)/k_v$ and $\tilde{k}_v(x)/\tilde{k}_v$. The definition of the level itself involves congruence conditions and so it is relatively easy to relate it to congruence conditions on V. After establishing the properties of the level, it is fairly straightforward to carry out the computation of $\bar{\varepsilon}_v(x)$.

For the rest of this introduction we discuss the organization of this part. Throughout this part, k is a fixed number field, and $\tilde{k}$ is a fixed quadratic extension of k. We also assume throughout that v is a dyadic place of k and $\tilde{k}_v$ is a ramified quadratic extension of k_v. Therefore, the content of this part is of a purely local nature. Even though we basically follow the notation and definitions in Part I, a minimal review of basic notions and definitions should help the reader, and we shall provide this in Section 2. In Section 3, we introduce the notion of the level of two ramified quadratic extension of a dyadic local field and establish its fundamental properties. For the sake of computing $\bar{\varepsilon}_v(x)$, Proposition 3.14 is the crucial result. In Section 4, we compute $\mathrm{vol}(G^{\circ}_{x\,k_v} \cap K_v)$ and prove that it depends only on the level of

$k_v(x)$ and $\tilde{k}_v$. In Section 5, we compute the sum of $\mathrm{vol}(K_v x)$ for each equivalence class of representatives, using the same method as that in Section 4 of Part II.

2. Review of facts from Part I

In this section we give a minimal review of basic notation and definitions from Part I that are needed in this part.

If X is a finite set then $\#X$ will denote its cardinality. The standard symbols $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Z}$ will denote respectively the rational, real and complex numbers and the rational integers. If $a \in \mathbb{R}$ then the largest integer z such that $z \leqslant a$ is denoted $\lfloor a \rfloor$ and the smallest integer z such that $z \geqslant a$ by $\lceil a \rceil$. If R is any ring then $R^\times$ is the set of invertible elements of R and if V is a variety defined over R then V_R denotes its R-points. If G is an algebraic group then G° denotes its identity component.

Throughout this paper, k is a fixed number field, $\tilde{k}$ is a fixed quadratic extension of k and v is a dyadic place of k such that $\tilde{k}_v = \tilde{k} \otimes_k k_v$ is a ramified quadratic extension of k_v. We denote the non-trivial element of $\mathrm{Gal}(\tilde{k}/k)$ by σ. Let $\mathcal{O}_v, \tilde{\mathcal{O}}_v$ be the integer rings of $k_v, \tilde{k}_v$ and $\mathfrak{p}_v = (\pi_v), \tilde{\mathfrak{p}}_v = (\tilde{\pi}_v)$ be their prime ideals. We denote the absolute value in k_v by $|\ |_v$. As far as notation pertaining to number fields and local fields, we use the same conventions as in Part I: the notation for the $\tilde{k}$ object will be derived from that of the k object by adding a tilde and, for other fields, by writing the field in question as the subscript. For example, $\mathcal{O}_F$ for the ring of integers of the field F. If $a \in k_v$ and $(a) = \mathfrak{p}_v^i$ then we write $\mathrm{ord}_{k_v}(a) = i$. If $\mathfrak{i}$ is a fractional ideal in k_v and $a - b \in \mathfrak{i}$ then we write $a \equiv b\ (\mathfrak{i})$ or $a \equiv b\ (c)$ if c generates $\mathfrak{i}$.

If k_1/k_2 is a finite extension either of local fields or of number fields then we shall write Δ_{k_1/k_2} for the relative discriminant of the extension; it is an ideal in the ring of integers of k_2. We put $\Delta_{\tilde{k}_v/k_v} = \mathfrak{p}_v^{\tilde{\delta}_v}$. We shall use the notation Tr_{k_1/k_2} and N_{k_1/k_2} for the trace and the norm in the extension k_1/k_2.

We assume that the reader is familiar with the basic definitions and facts concerning local fields. These may be found in [4]. We choose Haar measures dx_v on k_v and $d^\times t_v$ on $k_v^\times$ so that $\int_{\mathcal{O}_v} dx_v = 1$ and $\int_{\mathcal{O}_v^\times} d^\times t_v = 1$.

As in Part I, we use the following notation:

$$a(t_1, t_2) = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}, \quad n(u) = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}. \tag{2}$$

Let (G, V) be the prehomogeneous vector space (1) in the introduction. We identify $x = (x_1, x_2) \in V$ with the 2×2-matrix $M_x(v) = v_1 x_1 + v_2 x_2$ of linear forms in the variables v_1 and v_2, which we collect into the row vector $v = (v_1, v_2)$. With this identification, the action of $g = (g_1, g_2) \in G$ on V is $M_{gx}(v) = g_1 M_x(v g_2)^t g_1^\sigma$. We define $F_x(v) = -\det M_x(v)$. Then $F_{gx}(v) = \mathrm{N}_{\tilde{k}/k}(\det g_1) F_x(v g_2)$. It is proved in [1, p. 324] that, by associating x with the splitting field of $F_x(v)$, the orbit space $G_{k_v} \backslash V_{k_v}^{\mathrm{ss}}$ corresponds bijectively with field extensions F/k_v of degree one or two. If

$x \in V_{k_v}^{\mathrm{ss}}$ then we denote the corresponding field by $k_v(x)$. If $k_v(x) \neq k_v, \tilde{k}_v$ then we define $\tilde{k}_v(x)$ to be the compositum of $\tilde{k}_v$ and $k_v(x)$.

We use coordinate systems on G and V similar to those in Part I, as follows. For elements $g = (g_1, g_2) \in G$ we shall write

$$g_i = \begin{pmatrix} g_{i11} & g_{i12} \\ g_{i21} & g_{i22} \end{pmatrix} \tag{3}$$

for $i = 1, 2$. For vectors $x = (x_1, x_2) \in V$ we shall put

$$x_i = \begin{pmatrix} x_{i0} & x_{i1} \\ x_{i1}^{\sigma} & x_{i2} \end{pmatrix}. \tag{4}$$

With this coordinate system, $F_x(v) = a_0(x)v_1^2 + a_1(x)v_1v_2 + a_2(x)v_2^2$ where

$$\begin{aligned} a_0(x) &= \mathrm{N}_{\tilde{k}_v/k_v}(x_{11}) - x_{10}x_{12}, \\ a_1(x) &= \mathrm{Tr}_{\tilde{k}_v/k_v}(x_{11}x_{21}^{\sigma}) - x_{10}x_{22} - x_{12}x_{20}, \\ a_2(x) &= \mathrm{N}_{\tilde{k}_v/k_v}(x_{21}) - x_{20}x_{22}. \end{aligned} \tag{5}$$

Suppose that $p(z) = z^2 + a_1 z + a_2 \in k[z]$ has distinct roots α_1 and α_2. We collect these into a set $\alpha = \{\alpha_1, \alpha_2\}$, since the numbering is arbitrary. Define $w_p \in V_k$ by

$$w_p = \left(\begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix}, \begin{pmatrix} 1 & a_1 \\ a_1 & a_1^2 - a_2 \end{pmatrix} \right). \tag{6}$$

Then $F_{w_p}(z, 1) = p(z)$ and so we can choose a representative of the form w_p for each orbit in the orbit space $G_{k_v} \backslash V_{k_v}^{\mathrm{ss}}$. These are the *standard representatives*. As remarked in [3], (3.15) and what follows, if we put

$$w = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

and

$$h_\alpha = \begin{pmatrix} 1 & -1 \\ -\alpha_1 & \alpha_2 \end{pmatrix}$$

then we have $w_p = (h_\alpha, (\alpha_2 - \alpha_1)^{-1} h_\alpha) w$ if $k_v(w_p) \neq \tilde{k}_v$ and $w_p = (h_\alpha, h_\alpha, (\alpha_2 - \alpha_1)^{-1} h_\alpha) w$ if $k_v(w_p) = \tilde{k}_v$. (In the latter case we are regarding G_{k_v} as being embedded in $G_{\tilde{k}_v}$; this is explained more fully in [3], Section 3.)

We only consider x such that $k_v(x)/k_v$ is a ramified quadratic extension. Since $\tilde{k}_v/k_v$ is also ramified, by assumption, there are three types of orbits. By definition, the one corresponding to $\tilde{k}_v$ has index $(\mathrm{rm}\,\mathrm{rm})^*$, those corresponding to quadratic extensions $k_v(x)/k_v$ such that $k_v(x)\neq\tilde{k}_v$ and $\tilde{k}_v(x)/\tilde{k}_v$ is unramified have index $(\mathrm{rm}\,\mathrm{rm}\,\mathrm{ur})$ and those corresponding to quadratic extensions $k_v(x)/k_v$ such that $k_v(x)\neq\tilde{k}_v$ and $\tilde{k}_v(x)/\tilde{k}_v$ is ramified have index $(\mathrm{rm}\,\mathrm{rm}\,\mathrm{rm})$. These indices are used in Tables 1 and 2.

3. The level of paired quadratic fields

Let $k_1\neq k_2$ be ramified quadratic extensions of k_v, and $k_1\cdot k_2$ be the compositum of k_1 and k_2. We introduce the notion of the level and prove its fundamental properties in this section. For the rest of this paper we put $2\mathcal{O}_v=\mathfrak{p}_v^{m_v}$.

First we need to recall some facts concerning quadratic extensions of k_v. There is a unique unramified quadratic extension of k_v and it is well known that it is generated by a root of the Artin–Schreier polynomial z^2-z-c for a suitable choice of $c\in\mathcal{O}_v^\times$. Thus it is also generated by the square-root of $1+4c$. If $\varepsilon\in\mathcal{O}_v^\times$ is a unit whose square-root generates the unramified quadratic extension of k_v then $\varepsilon=a^2(1+4c)$ for some $a\in\mathcal{O}_v^\times$ and so the congruence $\varepsilon\equiv a^2\ (\mathfrak{p}_v^{2m_v})$ is solvable. Conversely, if $\varepsilon\in\mathcal{O}_v^\times$ is such that $\varepsilon\equiv a^2\ (\mathfrak{p}_v^{2m_v})$ is solvable but $\varepsilon\equiv a^2\ (\mathfrak{p}_v^{2m_v+1})$ is not, then ε is not a square and $(2a)^{-1}(a-\sqrt{\varepsilon})$ is easily seen to satisfy an Artin–Schreier polynomial, so that $\sqrt{\varepsilon}$ generates the unramified quadratic extension of k_v. Notice that $\varepsilon\equiv a^2\ (\mathfrak{p}_v^{2m_v+1})$ being solvable implies that ε is a square, by Hensel's lemma.

Now we turn to ramified quadratic extensions, F, of k_v. Every such extension is generated by a root of an Eisenstein polynomial $p(z)=z^2+a_1z+a_2$. This root is a uniformizer, π_F, of F and we have $\mathcal{O}_F=\mathcal{O}_v[\pi_F]$ and hence $\Delta_{F/k_v}=(a_1^2-4a_2)\mathcal{O}_v$ for any choice of Eisenstein polynomial which splits in F. If $\mathrm{ord}_{k_v}(a_1)\geqslant m_v+1$ then we may make the transformation $z\mapsto z-(a_1/2)$ in order to assume that $a_1=0$. These extensions are exactly those generated by the square-root of a uniformizer of k_v and they have $\Delta_{F/k_v}=\mathfrak{p}_v^{2m_v+1}$. If $1\leqslant\mathrm{ord}_{k_v}(a_1)\leqslant m_v$ then put $\ell=\mathrm{ord}_{k_v}(a_1)$. Here $\Delta_{F/k_v}=\mathfrak{p}_v^{2\ell}$ and F is generated by the square-root of $a_1^2-4a_2$ and hence also by the square-root of the unit $1-4a_2a_1^{-2}=1+\pi_v^{2(m_v-\ell)+1}c$ for a suitable $c\in\mathcal{O}_v^\times$. This exhausts all quadratic extensions of k_v. If $\varepsilon\in\mathcal{O}_v^\times$ is a non-square unit and $\varepsilon\equiv a^2\ (\mathfrak{p}_v^{2m_v})$ is not solvable then let $i<2m_v$ be the largest integer such that $\varepsilon\equiv a^2\ (\mathfrak{p}_v^{i})$ is solvable. We must have $\varepsilon=a^2(1+\pi_v^{2(m_v-\ell)+1}c)$ for some $1\leqslant\ell\leqslant m_v$ and $c\in\mathcal{O}_v^\times$ and then $i=2(m_v-\ell)+1$. In this case, $\pi_v^{\ell-m_v}(\sqrt{\varepsilon}-a)$ is a uniformizer of $k_v(\sqrt{\varepsilon})$. From this paragraph and the previous one it follows that if $\varepsilon\in\mathcal{O}_v^\times$ is a non-square unit then we may always multiply ε by a square to arrange either $\varepsilon=1+4c$ or $\varepsilon=1+\pi_v^{2(m_v-\ell)+1}c$ with $c\in\mathcal{O}_v^\times$.

In what follows we shall use the subscript 1 (resp. 2) to denote objects associated with k_1 (resp. k_2). Thus $\mathcal{O}_1$ will be the ring of integers of k_1, π_1 a uniformizer of k_1, $\mathfrak{p}_1$ the prime ideal in $\mathcal{O}_1$ and $\Delta_{k_1/k_v}=\mathfrak{p}_v^{\delta_1}$ and similarly with 1 replaced by 2. Let $p_1(z)=$

$z^2 + a_1 z + a_2$ and $p_2(z) = z^2 + b_1 z + b_2$ be the minimal polynomials of π_1 and π_2 over k_v, respectively. Let $\ell_1 = \mathrm{ord}_{k_v}(a_1)$ if this is less than or equal to m_v and $\ell_1 = m_v + 1$ otherwise. Define ℓ_2 similarly for k_2. Notice that we have $\ell_i = \lfloor (\delta_i + 1)/2 \rfloor$.

In the following two lemmas, F/k_v is a ramified quadratic extension, $\mathfrak{p}_F$ is the maximal ideal in the ring of integers of F and $\Delta_{F/k_v} = \mathfrak{p}_v^{\delta_F}$. We let

$$\ell_F = \lfloor (\delta_F + 1)/2 \rfloor.$$

Lemma 3.1. *Suppose* $x \in F$ *and* $\mathrm{ord}_F(x) = 1$. *Then* $\mathrm{Tr}_{F/k_v}(x) \in \mathfrak{p}_v^{\ell_F}$. *Moreover, if* $\ell_F \leqslant m_v$ *then* $\mathrm{ord}_{k_v}(\mathrm{Tr}_{F/k_v}(x)) = \ell_F$.

Proof. We have $\mathcal{O}_F = \mathcal{O}_v[x]$ and so if $z^2 + c_1 z + c_2$ is the minimal polynomial of x over k_v then $c_1 = -\mathrm{Tr}_{F/k_v}(x)$ and $(c_1^2 - 4c_2)\mathcal{O}_v = \Delta_{F/k_v}$. If $\ell_F \leqslant m_v$ then $\Delta_{F/k_v} = \mathfrak{p}_v^{2\ell_F}$ and hence $\mathrm{ord}_{k_v}(c_1) = \ell_F$. If $\ell_F = m_v + 1$ then $\Delta_{F/k_v} = \mathfrak{p}_v^{2m_v+1}$ and so $c_1^2 \in \mathfrak{p}_v^{2m_v+1}$, which gives $c_1 \in \mathfrak{p}_v^{\ell_F}$. $\square$

Lemma 3.2. *Suppose* $u \in F$ *and* $\mathrm{ord}_F(u) = j$. *Then*

$$\mathrm{ord}_{k_v}(\mathrm{Tr}_{F/k_v}(u)) \geqslant \lfloor (j + \delta_F)/2 \rfloor.$$

Proof. The different of F/k_v is $\mathfrak{p}_F^{\delta_F}$ and so, from the definition of the different, $u \in \mathfrak{p}_F^{-\delta_F}$ implies that $\mathrm{Tr}_{F/k_v}(u) \in \mathcal{O}_v$. Multiplying by π_v^n, we find that $u \in \mathfrak{p}_F^{2n-\delta_F}$ implies that $\mathrm{Tr}_{F/k_v}(u) \in \mathfrak{p}_v^n$. Let $n = \lfloor (j + \delta_F)/2 \rfloor$. Then $2n \leqslant j + \delta_F$ and so $2n - \delta_F \leqslant j$. Thus $u \in \mathfrak{p}_F^{2n-\delta_F}$ and so $\mathrm{Tr}_{F/k_v}(u) \in \mathfrak{p}_v^n$. $\square$

For $0 \leqslant i_1 \leqslant i_2 \leqslant i_1 + 1$ we define

$$S_{i_1,i_2}(k_1,k_2) = \left\{ \eta \in \mathcal{O}_2/\pi_2^{i_1+i_2}\mathcal{O}_2 \;\middle|\; \begin{array}{l} \mathrm{Tr}_{k_2/k_v}(\eta) \equiv a_1 \ (\mathfrak{p}_v^{i_1}), \\ \mathrm{N}_{k_2/k_v}(\eta) \equiv a_2 \ (\mathfrak{p}_v^{i_2}) \end{array} \right\}. \tag{7}$$

We first show that the conditions defining $S_{i_1,i_2}(k_1,k_2)$ depend only on the class of η modulo $\pi_2^{i_1+i_2}\mathcal{O}_2$, so that the definition makes sense. Suppose that $\eta \in \mathcal{O}_2$ and $u \in \pi_2^{i_1+i_2}\mathcal{O}_2$. Then, by Lemma 3.2,

$$\mathrm{ord}_{k_v}(\mathrm{Tr}_{k_2/k_v}(u)) \geqslant \lfloor (i_1 + i_2 + \delta_2)/2 \rfloor \geqslant \lfloor (2i_1 + \delta_2)/2 \rfloor \geqslant i_1$$

and so $\mathrm{Tr}_{k_2/k_v}(\eta) \equiv \mathrm{Tr}_{k_2/k_v}(\eta + u) \ (\mathfrak{p}_v^{i_1})$. Also,

$$\mathrm{N}_{k_2/k_v}(\eta + u) = \mathrm{N}_{k_2/k_v}(\eta) + \mathrm{Tr}_{k_2/k_v}(\eta^\sigma u) + \mathrm{N}_{k_2/k_v}(u)$$

and

$$\mathrm{ord}_{k_v}(\mathrm{Tr}_{k_2/k_v}(\eta^\sigma u)) \geqslant \lfloor (i_1 + i_2 + \delta_2)/2 \rfloor \geqslant \lfloor (2i_2 + \delta_2 - 1)/2 \rfloor \geqslant i_2$$

by Lemma 3.2 and the fact that $\delta_2 \geqslant 2$. Further, $\mathrm{ord}_{k_v}(\mathrm{N}_{k_2/k_v}(u)) = i_1 + i_2 \geqslant i_2$ and so $\mathrm{N}_{k_2/k_v}(\eta + u) \equiv \mathrm{N}_{k_2/k_v}(\eta)\ (\mathfrak{p}_v^{i_2})$. We shall, by a slight abuse of notation, confuse elements of $\mathcal{O}_2$ with their classes modulo $\pi_2^{i_1+i_2}\mathcal{O}_2$, so that we may write $\eta \in S_{i_1,i_2}(k_1,k_2)$ if the class of $\eta \in \mathcal{O}_2$ satisfies the indicated conditions.

We let $\mathfrak{n}_1(k_1,k_2,i)$ (resp. $\mathfrak{n}_2(k_1,k_2,i)$) be the cardinality of the set $S_{i,i}(k_1,k_2)$ (resp. $S_{i,i+1}(k_1,k_2)$) for $i \geqslant 0$. The set $S_{i_1,i_2}(k_1,k_2)$ depends on the choice of an Eisenstein polynomial for k_1. However, it is $\mathfrak{n}_1(k_1,k_2,i)$ and $\mathfrak{n}_2(k_1,k_2,i)$ which interest us and it turns out that these numbers depend only on k_1, k_2 and i, as we show in Lemma 3.4 below. In fact, we are really only interested in the range of i in which $\mathfrak{n}_1(k_1,k_2,i)$ and $\mathfrak{n}_2(k_1,k_2,i)$ do not vanish and Lemma 3.4 is more than we require. We shall discuss the motivation for our approach at the end of this section.

Definition 3.3. The largest integer i such that $S_{i,i}(k_1,k_2) \neq \emptyset$ will be called the level of k_1 and k_2 and denoted by $\mathrm{lev}(k_1,k_2)$.

Of course, $\mathrm{lev}(k_1,k_2)$ is the largest integer, i, such that $\mathfrak{n}_1(k_1,k_2,i) \neq 0$. It is an easy consequence of Hensel's lemma that $\mathrm{lev}(k_1,k_2) < \infty$, since k_1 and k_2 are distinct. A specific upper bound for $\mathrm{lev}(k_1,k_2)$ will be given in Proposition 3.8. It follows directly from the definition that

$$\mathfrak{n}_1(k_1,k_2,0) = \mathfrak{n}_2(k_1,k_2,0) = 1, \quad \mathfrak{n}_1(k_1,k_2,1) = q_v. \tag{8}$$

Lemma 3.4. (3.4.1) *The numbers* $\mathfrak{n}_1(k_1,k_2,i)$ *and* $\mathfrak{n}_2(k_1,k_2,i)$ *depend only on* k_1 *and* k_2, *not on the particular choice of Eisenstein polynomial used to evaluate them. Thus this notation is legitimate.*

(3.4.2) *For* $j = 1, 2$, *we have* $\mathfrak{n}_j(k_2,k_1,i) = \mathfrak{n}_j(k_1,k_2,i)$ *for all* $i \geqslant 0$.

Proof. If π_1 and π_1' are uniformizers of k_1 then $\pi_1 = c + d\pi_1'$ with $c \in \mathfrak{p}_v$ and $d \in \mathcal{O}_v^\times$. If $p_1(z) = z^2 + a_1 z + a_2$ is the Eisenstein polynomial associated to π_1 then the Eisenstein polynomial, $p_1'(z) = z^2 + a_1' z + a_2'$, associated to π_1' is $p_1'(z) = z^2 + d^{-1}(a_1 + 2c)z + d^{-2}(c^2 + a_1 c + a_2)$. Say $\eta \in \mathcal{O}_2$ satisfies the congruences $\mathrm{Tr}_{k_2/k_v}(\eta) \equiv a_1\ (\mathfrak{p}_v^{i_1})$ and $\mathrm{N}_{k_2/k_v}(\eta) \equiv a_2\ (\mathfrak{p}_v^{i_2})$. Then it is easy to check that $\eta' = d^{-1}(\eta + c)$ satisfies the congruences $\mathrm{Tr}_{k_2/k_v}(\eta') \equiv a_1'\ (\mathfrak{p}_v^{i_1})$ and $\mathrm{N}_{k_2/k_v}(\eta') \equiv a_2'\ (\mathfrak{p}_v^{i_2})$. Since $d \in \mathcal{O}_v^\times$, the map $\eta \mapsto d^{-1}(\eta + c)$ induces a well-defined map on $\mathcal{O}_2/\pi_2^{i_1+i_2}\mathcal{O}_2$ with inverse induced by $\eta' \mapsto d\eta' - c$. This establishes a one-to-one correspondence between the two sets and (3.4.1) follows.

Fix a uniformizer π_1 of k_1 and let $p_1(z) = z^2 + a_1 z + a_2$ be the corresponding Eisenstein polynomial. Consider $S_{i_1,i_2}(k_1,k_2)$. We may assume that $i_2 \geqslant 2$, since we have evaluated the numbers $\mathfrak{n}_1(k_1,k_2,0)$, $\mathfrak{n}_1(k_1,k_2,1)$ and $\mathfrak{n}_2(k_1,k_2,0)$ in (8) and they satisfy the second claim. With this assumption, every element of $S_{i_1,i_2}(k_1,k_2)$ is (the class of) a uniformizer in $\mathcal{O}_2$.

Suppose $S_{i_1,i_2}(k_1,k_2)\neq\emptyset$. Fix $\eta_0\in S_{i_1,i_2}(k_1,k_2)$. We will use the corresponding Eisenstein polynomial $p_0(z)=z^2+a_{01}z+a_{02}$ to evaluate $\mathfrak{n}_j(k_2,k_1,i)$. Every other element η of $S_{i_1,i_2}(k_1,k_2)$ has the form $\eta=c(\eta)+d(\eta)\eta_0$ with $c(\eta)\in\mathfrak{p}_v$ and $d(\eta)\in\mathcal{O}_v^\times$. Moreover, the conditions on η imply that $c(\eta)$ and $d(\eta)$ satisfy the congruences

$$-d(\eta)(a_{01}-2c(\eta)d(\eta)^{-1})\equiv a_1 \ (\mathfrak{p}_v^{i_1}),$$

$$c(\eta)^2-a_{01}c(\eta)d(\eta)+a_{02}d(\eta)^2\equiv a_2 \ (\mathfrak{p}_v^{i_2}).$$

We define $\varpi(\eta)=d(\eta)^{-1}(\pi_1+c(\eta))$. Then, using the facts that $c(\eta)\in\mathfrak{p}_v$ and $i_2\leqslant i_1+1$, it is easy to check that

$$\mathrm{Tr}_{k_1/k_v}(\varpi(\eta))\equiv a_{01} \ (\mathfrak{p}_v^{i_1}),\ \ \mathrm{N}_{k_1/k_v}(\varpi(\eta))\equiv a_{02} \ (\mathfrak{p}_v^{i_2})$$

and so $\varpi(\eta)\in S_{i_1,i_2}(k_2,k_1)$. Suppose $u\in\pi_2^{i_1+i_2}\mathcal{O}_2$ and $\eta'=\eta+u$. If we write $u=c(u)+d(u)\eta_0$ with $d(c),d(u)\in\mathcal{O}_v$ then $c(u)\in\mathfrak{p}_v^{i_2}$ and $d(u)\in\mathfrak{p}_v^{i_1}$. By computation,

$$\begin{aligned}&d(\eta')^{-1}(\pi_1+c(\eta'))-d(\eta)^{-1}(\pi_1+c(\eta))\\&\quad=d(\eta')^{-1}d(\eta)^{-1}(-d(u)\pi_1+c(u)d(\eta)-c(\eta)d(u)).\end{aligned}$$

It is easy to check that this element belongs to $\pi_1^{i_1+i_2}\mathcal{O}_1$ and so the map $\eta\mapsto\varpi(\eta)$ induces a well-defined map from $S_{i_1,i_2}(k_1,k_2)$ to $S_{i_1,i_2}(k_2,k_1)$. Reversing the roles of k_1 and k_2 we obtain a similar map from $S_{i_1,i_2}(k_2,k_1)$ to $S_{i_1,i_2}(k_1,k_2)$ induced by the map sending $\zeta=c'(\zeta)+d'(\zeta)\pi_1$ to $d'(\zeta)^{-1}(\eta_0+c'(\zeta))$. It is easy to check that these maps are inverse to one another and so $S_{i_1,i_2}(k_1,k_2)$ and $S_{i_1,i_2}(k_2,k_1)$ have the same cardinality. $\square$

Let k_3 be the unique quadratic extension of k_v other than k_1 and k_2 contained in $k_1\cdot k_2$. Let $p_1(z),p_2(z)$ be as before. Let α_1 and α_2 be the roots of p_1 and β_1 and β_2 be the roots of p_2.

Define

$$\begin{aligned}\gamma_1&=(\alpha_1-\beta_1)(\alpha_2-\beta_2),\\ \gamma_2&=(\alpha_1-\beta_2)(\alpha_2-\beta_1).\end{aligned}$$

The following lemma provides an equation defining k_3. We will not provide the proof, since it is elementary.

Lemma 3.5. *The numbers γ_1 and γ_2 generate k_3 over k_v and are the roots of the polynomial*

$$p_3(z)=z^2-[2(a_2+b_2)-a_1b_1]z+J, \tag{9}$$

where $J = (a_2 - b_2)^2 + (a_1 - b_1)(a_1 b_2 - a_2 b_1)$. *Moreover*, $\gamma_1 - \gamma_2 = (\alpha_1 - \alpha_2)(\beta_1 - \beta_2)$.

Next we consider the relation between discriminants of k_1, k_2, k_3. Let $\Delta_{k_i/k_v} = \mathfrak{p}_v^{\delta_i}$ for $i = 1, 2, 3$. Note that for $i = 1, 2$, $\delta_i = 2, \ldots, 2m_v$ or $2m_v + 1$.

Lemma 3.6. *We have* $\delta_3 \leqslant \max\{\delta_1, \delta_2\}$. *Moreover, equality holds if* $\delta_1 \neq \delta_2$.

Proof. There are two cases to consider. If two of the fields are generated by adjoining the square-root of a uniformizer then they have equal discriminants and the third field has a smaller discriminant (since it is obtained by adjoining the square root of a unit). Therefore, we have the statement of this lemma in this case. Otherwise, all the fields are obtained by adjoining the square root of a unit. Let ε_1, ε_2 and ε_3 be the units whose square roots generate k_1, k_2 and k_3 respectively. We may assume that $\varepsilon_j = 1 + \pi_v^{2(m_v - \ell_j)+1} c_j$ where $c_j \in \mathcal{O}_v^\times$ and $\delta_j = 2\ell_j$ for $j = 1, 2$. We may also assume $\varepsilon_3 = \varepsilon_1 \varepsilon_2$. Then

$$\varepsilon_3 = 1 + \pi_v^{2(m_v - \ell_1)+1} c_1 + \pi_v^{2(m_v - \ell_2)+1} c_2 + \pi_v^{2(m_v - \ell_1)+2(m_v - \ell_2)+2} c_1 c_2.$$

If $\ell_1 > \ell_2$ then

$$\varepsilon_3 \equiv \varepsilon_1 \ \left(\mathfrak{p}_v^{2(m_v - \ell_1)+2}\right)$$

and so $\delta_3 = 2\ell_1 = \delta_1$. The case $\ell_2 > \ell_1$ is similar. If $\ell_1 = \ell_2$ then $\varepsilon_3 \equiv 1 \ (\mathfrak{p}_v^{2(m_v - \ell_1)+1})$. If $\varepsilon_3 \equiv 1 \ (4)$ then $\delta_3 = 0$ and the inequality holds true. Otherwise, the largest number, i, such that $\varepsilon_3 \equiv 1 \ (\mathfrak{p}_v^i)$ has the form $i = 2(m_v - \ell_3) + 1$ with $\ell_3 \leqslant \ell_1 = \ell_2$. Then $\delta_3 = 2\ell_3 \leqslant \delta_1 = \delta_2$ and again the inequality is true. □

Lemma 3.7. *We have*

$$2\lfloor \tfrac{1}{2} \operatorname{ord}_{k_v}(J) \rfloor \leqslant \delta_1 + \delta_2 - \delta_3.$$

Proof. Let $a = \lfloor \frac{1}{2} \operatorname{ord}_{k_v}(J) \rfloor$ so that J/π_v^{2a} is either a unit of a uniformizer of k_v. Since $\mathrm{N}_{k_3/k_v}(\gamma_j/\pi_v^a) = J/\pi_v^{2a}$ for $j = 1$ and 2, we conclude that γ_j/π_v^a is an integer. Thus the ideal generated by $(\gamma_1 - \gamma_2)^2/\pi_v^{2a}$ in $\mathcal{O}_v$ is contained in $\mathfrak{p}_v^{\delta_3}$. But $\gamma_1 - \gamma_2 = (\alpha_1 - \alpha_2)(\beta_1 - \beta_2)$ and so the ideal generated by $(\gamma_1 - \gamma_2)^2$ is $\mathfrak{p}_v^{\delta_1 + \delta_2}$. The inequality follows. □

Proposition 3.8. *We have* $2\,\mathrm{lev}(k_1, k_2) + \delta_3 \leqslant \delta_1 + \delta_2$.

Proof. Let $i = \mathrm{lev}(k_1, k_2)$. We choose Eisenstein polynomials p_1, p_2 so that $a_1 \equiv b_1$, $a_2 \equiv b_2 \ (\mathfrak{p}_v^i)$. Then

$$J = (a_2 - b_2)^2 + (a_1 - b_1)[a_1(b_2 - a_2) + a_2(a_1 - b_1)] \tag{10}$$

and our assumptions imply that this lies in $\mathfrak{p}_v^{2i}$. Using the previous lemma we obtain $2i \leqslant \delta_1 + \delta_2 - \delta_3$ and the inequality follows. □

Corollary 3.9. (3.9.1) *If* $\ell_1 \neq \ell_2$ *then* $\mathrm{lev}(k_1, k_2) \leqslant \min\{\ell_1, \ell_2\}$.
(3.9.2) *If* $\ell_1 = \ell_2 = \ell$, $\delta_1 = \delta_2 = \delta$ *then* $\mathrm{lev}(k_1, k_2) \leqslant \delta$.

Proof. Consider (3.9.1) Suppose, without loss of generality, that $\ell_1 < \ell_2$. Then, according to Lemma 3.6, we must have $\delta_3 = \delta_2$ and so the inequality in Proposition 3.8 becomes $\mathrm{lev}(k_1, k_2) \leqslant \frac{1}{2}\delta_1$. Since $\delta_1 \neq 2m_v + 1$, $\frac{1}{2}\delta_1 = \ell_1$. Statement (3.9.2) is obvious from Proposition 3.8 because $\delta_3 \geqslant 0$. □

Note the above corollary implies that if $k_1, k_2/k_v$ are ramified quadratic extensions, $\delta_1 = \delta_2 = \delta$ and $S_{\delta+1,\delta+1}(k_1, k_2) \neq \emptyset$ then $k_1 = k_2$.

Proposition 3.10. *The extension* $(k_1 \cdot k_2)/k_2$ *is unramified if and only if* $\delta_1 = \delta_2$ *and* $S_{\delta_1,\delta_1}(k_1, k_2) \neq \emptyset$. *Moreover, if these conditions are satisfied then* k_3/k_v *is unramified.*

Proof. Suppose $k_1 = k_v(\sqrt{\varepsilon_1})$ and $k_2 = k_v(\sqrt{\varepsilon_2})$. We first assume $(k_1 \cdot k_2)/k_2$ is unramified. Then $(k_1 \cdot k_2)/k_v$ is not totally ramified. Therefore, by [4, Corollary 4, p. 19], $k_1 \cdot k_2$ contains an unramified quadratic extension of k_v. Since k_1, k_2 are ramified, the remaining quadratic subfield $k_3 = k_v(\sqrt{\varepsilon_2\varepsilon_1^{-1}})$ must be unramified over k_v. Let $\varepsilon_3 = \varepsilon_2\varepsilon_1^{-1}$, so that $\varepsilon_2 = \varepsilon_1\varepsilon_3$. Multiplying ε_2 and hence ε_3 by a square, if necessary, we may assume that $\varepsilon_3 \equiv 1$ (4). Then ε_1 and ε_2 have the same order in k_v and, multiplying them both by the same square, we may assume that they are either both units or both uniformizers without altering ε_3.

If $\varepsilon_1, \varepsilon_2$ are both uniformizers then $\delta_1 = \delta_2 = 2m_v + 1$. By the assumption on ε_3, $\varepsilon_2 = \varepsilon_1(1 + 4c_3)$ for some $c_3 \in \mathcal{O}_v^\times$. Let $\eta_i = \sqrt{\varepsilon_i}$ for $i = 1, 2$. Then η_1, η_2 are uniformizers of k_1, k_2, respectively, and

$$\mathrm{Tr}_{k_2/k_v}(\eta_2) = \mathrm{Tr}_{k_1/k_v}(\eta_1) = 0,$$

$$\mathrm{N}_{k_2/k_v}(\eta_2) = -\varepsilon_2 = -\varepsilon_1 - 4\varepsilon_1 c_3 \equiv \mathrm{N}_{k_1/k_v}(\eta_1) \ \ (\mathfrak{p}_v^{2m_v+1}).$$

This implies that $S_{2m_v+1,2m_v+1}(k_1, k_2) \neq \emptyset$.

Suppose $\varepsilon_1, \varepsilon_2$ are both units. Then $\delta_1 = 2\ell_1, \delta_2 = 2\ell_2$ with $1 \leqslant \ell_1$, $\ell_2 \leqslant m_v$. Let $\varepsilon_1 = 1 + \pi_v^{2(m_v-\ell_1)+1}c_1$ and $\varepsilon_3 = 1 + 4c_3$ with $c_1 \in \mathcal{O}_v^\times$, $c_3 \in \mathcal{O}_v^\times$. Then

$$\varepsilon_2 = \varepsilon_1\varepsilon_3 = 1 + \pi_v^{2(m_v-\ell_1)+1}(c_1 + (4\pi_v^{-2m_v})\pi_v^{2\ell_1-1}c_3 + 4c_1c_3).$$

Let

$$c_2 = c_1 + (4\pi_v^{-2m_v})\pi_v^{2\ell_1-1}c_3 + 4c_1c_3. \tag{11}$$

Then $c_2 \in \mathcal{O}_v^\times$, $\varepsilon_2 = 1 + \pi_v^{2(m_v-\ell_1)+1}c_2$ and $c_2 \equiv c_1$ $(\mathfrak{p}_v^{2\ell_1-1})$. Therefore, $\ell_1 = \ell_2$ and so $\delta_1 = \delta_2$.

Let $\delta = \delta_1 = \delta_2$ and $\ell = \ell_1 = \ell_2$. We put $\eta_i = (\pi_v^\ell/2)(\sqrt{\varepsilon_i} - 1)$ for $i = 1, 2$. Then η_i is a uniformizer of k_i satisfying the Eisenstein equation $z^2 + \pi_v^\ell z - \pi_v \theta c_i = 0$ for $i = 1, 2$ where $\theta = \pi_v^{2m_v}/4 \in \mathcal{O}_v^\times$. Thus

$$\mathrm{Tr}_{k_2/k_v}(\eta_2) = -\pi_v^\ell = \mathrm{Tr}_{k_1/k_v}(\eta_1),$$

$$\mathrm{N}_{k_2/k_v}(\eta_2) = -\pi_v \theta c_2 \equiv -\pi_v \theta c_1 \ (\mathfrak{p}_v^{2\ell})$$

and since $\mathrm{N}_{k_1/k_v}(\eta_1) = -\pi_v \theta c_1$, we have $\eta_2 \in S_{\delta,\delta}(k_1, k_2)$.

Conversely, suppose $\delta_1 = \delta_2$ and $S_{\delta_1,\delta_1}(k_1, k_2) \neq \emptyset$. Let k_3, δ_3 be as before. Then, by Proposition 3.8, $\delta_3 = 0$. This implies that k_3/k_v is unramified. Since k_3 is generated by roots of an Artin–Schreier equation and they also generate the field extension $(k_1 \cdot k_2)/k_2$, this extension is unramified also. $\square$

Note that by Proposition 3.10, there is precisely one orbit whose index is (rm rm ur).

We shall next prove that $\mathrm{lev}(k_1, k_2) \geqslant \min\{\lfloor \frac{1}{2}(\delta_1 + 1) \rfloor, \lfloor \frac{1}{2}(\delta_2 + 1) \rfloor\}$.

Lemma 3.11. *Suppose that $\ell_1 \neq \ell_2$ and $1 \leqslant i \leqslant \min\{\ell_1, \ell_2\}$ or that $1 \leqslant i < \ell = \ell_1 = \ell_2$. If $\eta \in \mathcal{O}_2$ satisfies $\mathrm{ord}_{k_2}(\eta) = 1$ and $\mathrm{N}_{k_2/k_v}(\eta) \equiv a_2 \ (\mathfrak{p}_v^i)$ then there exists a unit $t = c - d\eta$ such that $\mathrm{N}_{k_2/k_v}(t\eta) \equiv a_2 \ (\mathfrak{p}_v^{i+1})$.*

Proof. If $i = 1$, we choose $t \in \mathcal{O}_v^\times$. Then $\mathrm{N}_{k_2/k_v}(t\eta) = t^2 \mathrm{N}_{k_2/k_v}(\eta)$. Since any element of $\mathcal{O}_v^\times$ is a square modulo $\mathfrak{p}_v$, we can choose t so that $t^2 \mathrm{N}_{k_2/k_v}(\eta) \equiv a_2 \ (\mathfrak{p}_v^2)$.

We now assume $i \geqslant 2$. Note that if $\ell_1 \neq \ell_2$ then $\ell_1 \leqslant m_v$ or $\ell_2 \leqslant m_v$ and so $i + 1 \leqslant m_v + 1$. This condition is obviously satisfied in the second case.

Suppose $\eta^2 + b_1'\eta + b_2' = 0$ is the Eisenstein equation satisfied by η. Let $\mathrm{N}_{k_2/k_v}(\eta) = b_2' = a_2 + e\pi_v^i$. Then

$$\begin{aligned} \mathrm{N}_{k_2/k_v}(t\eta) &= \mathrm{N}_{k_2/k_v}(t)\mathrm{N}_{k_2/k_v}(\eta) \\ &= (c^2 + b_1'cd + b_2'd^2)(a_2 + e\pi_v^i) \\ &\equiv (c^2 + b_2'd^2)(a_2 + e\pi_v^i) \ (\mathfrak{p}_v^{i+1}) \\ &\equiv a_2c^2 + a_2b_2'd^2 + c^2e\pi_v^i \ (\mathfrak{p}_v^{i+1}). \end{aligned}$$

Note that, since $i \leqslant \ell_2$ in both cases, $b_1'\pi_v \equiv 0 \ (\mathfrak{p}_v^{i+1})$. Let $c = 1 + \pi_v^N f$ with $N > 0$ and $f \in \mathcal{O}_v^\times$. Then

$$c^2 = 1 + 2\pi_v^N f + \pi_v^{2N} f^2 \equiv 1 + \pi_v^{2N} f^2 \ (\mathfrak{p}_v^{i+1}).$$

The last congruence is satisfied because of the condition $i + 1 \leqslant m_v + 1$.

So

$$\begin{aligned}\mathrm{N}_{k_2/k_v}(t\eta) &\equiv a_2 + a_2 f^2\pi_v^{2N} + a_2 b_2' d^2 + e\pi_v^i + \pi_v^{i+2N} e f^2 \\ &\equiv a_2 + a_2 f^2 \pi_v^{2N} + a_2 b_2' d^2 + e\pi_v^i \ (\mathfrak{p}_v^{i+1}).\end{aligned}$$

Note that the orders of $a_2 f^2 \pi_v^{2N}, a_2 b_2' d^2$ are odd and even, respectively, and they can be any odd or even integer greater than or equal to two. So we can choose suitable d, f, N so that

$$a_2 f^2 \pi_v^{2N} + a_2 b_2' d^2 + e\pi_v^i \equiv 0 \ (\mathfrak{p}_v^{i+1}). \qquad \square$$

The following proposition provides a lower bound for $\mathrm{lev}(k_1, k_2)$.

Proposition 3.12. *Suppose* $1 \leqslant i \leqslant \min\{\ell_1, \ell_2\}$. *Then* $S_{i,i}(k_1, k_2) \neq \emptyset$ *and so* $\mathrm{lev}(k_1, k_2) \geqslant \min\{\ell_1, \ell_2\}$. *Moreover, if* $\ell_1 \neq \ell_2$ *then* $\mathrm{lev}(k_1, k_2) = \ell = \min\{\ell_1, \ell_2\}$ *and* $S_{\ell,\ell+1}(k_1, k_2) \neq \emptyset$.

Proof. We put $\ell = \min\{\ell_1, \ell_2\}$. Let $\eta \in \mathcal{O}_2$ be any uniformizer. Using Lemma 3.11 we can arrange that $\mathrm{N}_{k_2/k_v}(\eta) \equiv a_2 \ (\mathfrak{p}_v^{\ell})$ and $\mathrm{N}_{k_2/k_v}(\eta) \equiv a_2 \ (\mathfrak{p}_v^{\ell+1})$ if $\ell_1 \neq \ell_2$. By Lemma 3.1, $\mathrm{Tr}_{k_2/k_v}(\eta) \in \mathfrak{p}_v^{\ell_2} \subset \mathfrak{p}_v^{\ell}$ and so $\mathrm{Tr}_{k_2/k_v}(\eta) \equiv a_1 \ (\mathfrak{p}_v^{\ell})$. Thus $\eta \in S_{\ell,\ell}(k_1, k_2)$ and $\eta \in S_{\ell,\ell+1}(k_1, k_2)$ if $\ell_1 \neq \ell_2$. When $\ell_1 \neq \ell_2$, the equality $\mathrm{lev}(k_1, k_2) = \ell$ then follows from Corollary 3.9. $\square$

Note that $\ell_i = \lfloor \frac{1}{2}(\delta_i + 1) \rfloor$ for $i = 1, 2$ and so the above lower bound is $\min\{\lfloor \frac{1}{2}(\delta_1 + 1) \rfloor, \lfloor \frac{1}{2}(\delta_2 + 1) \rfloor\}$.

Lemma 3.13. *Suppose* $i \geqslant 1$, $\eta, \eta' \in \mathcal{O}_2$, *and* $\mathrm{N}_{k_2/k_v}(\eta) \equiv \mathrm{N}_{k_2/k_v}(\eta') \equiv a_2 \ (\mathfrak{p}_v^{i+1})$. *Then there exist* $e, f \in \mathcal{O}_v$ *such that* $\eta' = e\pi_v + (1 + f\pi_v)\eta$.

Proof. Note that η, η' are both uniformizers. So we may assume that $\eta' = c + d\eta$ with $c \in \mathfrak{p}_v, d \in \mathcal{O}_v^{\times}$. Then $\mathrm{N}_{k_2/k_v}(\eta') \equiv d^2 \mathrm{N}_{k_2/k_v}(\eta) \equiv d^2 a_2 \ (\mathfrak{p}_v^2)$. Therefore, $d^2 \equiv 1 \ (\mathfrak{p}_v)$. This implies $d \equiv 1 \ (\mathfrak{p}_v)$. $\square$

In the following proposition and its corollary we assume that $\ell_1 = \ell_2 = \ell$, $\delta_1 = \delta_2 = \delta$ and $\ell \leqslant i < \delta$.

Proposition 3.14. (3.14.1) *Suppose* $\eta \in \mathcal{O}_2$ *satisfies* $\mathrm{Tr}_{k_2/k_v}(\eta) \equiv a_1 \ (\mathfrak{p}_v^i)$ *and* $\mathrm{N}_{k_2/k_v}(\eta) \equiv a_2 \ (\mathfrak{p}_v^i)$. *Then there exists an* $\eta' \in \mathcal{O}_2$ *such that* $\mathrm{Tr}_{k_2/k_v}(\eta') \equiv a_1 \ (\mathfrak{p}_v^i)$ *and* $\mathrm{N}_{k_2/k_v}(\eta') \equiv a_2 \ (\mathfrak{p}_v^{i+1})$.

(3.14.2) *Suppose* $\eta \in \mathcal{O}_2$ *satisfies* $\mathrm{ord}_{k_v}(\mathrm{Tr}_{k_2/k_v}(\eta) - a_1) = i$, $\mathrm{N}_{k_2/k_v}(\eta) \equiv a_2 \ (\mathfrak{p}_v^{i+1})$. *If* $\eta' \in \mathcal{O}_2$, $\mathrm{Tr}_{k_2/k_v}(\eta') \equiv a_1 \ (\mathfrak{p}_v^i)$ *and* $\mathrm{N}_{k_2/k_v}(\eta') \equiv a_2 \ (\mathfrak{p}_v^{i+1})$, *we have* $\mathrm{ord}_{k_v}(\mathrm{Tr}_{k_2/k_v}(\eta') - a_1) = i$.

Proof. We first consider the case $i = \ell = 1$. For any uniformizer $\eta \in \mathcal{O}_2$, $\mathrm{Tr}_{k_2/k_v}(\eta) \equiv a_1 \equiv 0 \ (\mathfrak{p}_v)$. So the statement (3.14.1) follows from the fact that any unit in $\mathcal{O}_v$ is a square modulo $\mathfrak{p}_v$. Consider (3.14.2). By Lemma 3.13, there exist $e, f \in \mathcal{O}_v$ such that $\eta' = e\pi_v + (1 + f\pi_v)\eta$. Let $\mathrm{Tr}_{k_2/k_v}(\eta) = a_1 + h\pi_v$ with $h \in \mathcal{O}_v^\times$. Then

$$\begin{aligned}\mathrm{Tr}_{k_2/k_v}(\eta') &= 2\pi_v e + (1 + f\pi_v)\mathrm{Tr}_{k_2/k_v}(\eta)\\ &= 2\pi_v e + (1 + f\pi_v)(a_1 + h\pi_v)\\ &\equiv a_1 + h\pi_v \ (\mathfrak{p}_v^2).\end{aligned}$$

So $\mathrm{ord}_{k_v}(\mathrm{Tr}_{k_2/k_v}(\eta') - a_1) = 1$ also. This proves the proposition when $i = \ell = 1$.

Suppose $i \geqslant 2$ and $\eta \in S_{i,i}(k_1, k_2)$. Let $\mathrm{Tr}_{k_2/k_v}(\eta) - a_1 = \gamma_1\pi_v^i$ and $\mathrm{N}_{k_2/k_v}(\eta) - a_2 = \gamma_2\pi_v^i$, where $\gamma_i \in \mathcal{O}_2$ for $i = 1, 2$. For (3.14.1), we look for an element of the form $\eta' = e\pi_v + (1 + f\pi_v)\eta$ with $e, f \in \mathcal{O}_v$. If η' satisfies the condition of (3.14.2), η' is of the above form by Lemma 3.13. Therefore, in both cases we consider η' of the above form. Then

$$\mathrm{Tr}_{k_2/k_v}(\eta') = 2e\pi_v + (1 + f\pi_v)(a_1 + \gamma_1\pi_v^i),$$

$$\begin{aligned}\mathrm{N}_{k_2/k_v}(\eta') &= e^2\pi_v^2 + e\pi_v(1 + f\pi_v)(a_1 + \gamma_1\pi_v^i)\\ &\quad + (1 + f\pi_v)^2(a_2 + \gamma_2\pi_v^i).\end{aligned}$$

So

$$\mathrm{Tr}_{k_2/k_v}(\eta') - a_1 \equiv 2e\pi_v + a_1 f\pi_v + \gamma_1\pi_v^i \ (\mathfrak{p}_v^{i+1}),$$

$$\begin{aligned}\mathrm{N}_{k_2/k_v}(\eta') - a_2 &\equiv e^2\pi_v^2 + a_1 e\pi_v + a_1 ef\pi_v^2\\ &\quad + 2a_2 f\pi_v + a_2 f^2\pi_v^2 + \gamma_2\pi_v^i \ (\mathfrak{p}_v^{i+1}).\end{aligned} \tag{12}$$

Consider the case $2 \leqslant i = \ell \leqslant m_v$. We have

$$\begin{aligned}&\mathrm{Tr}_{k_2/k_v}(\eta') - a_1 \equiv \gamma_1\pi_v^i \ (\mathfrak{p}_v^{i+1}),\\ &\mathrm{N}_{k_2/k_v}(\eta') - a_2 \equiv e^2\pi_v^2 + a_2 f^2\pi_v^2 + \gamma_2\pi_v^i \ (\mathfrak{p}_v^{i+1}).\end{aligned} \tag{13}$$

Since the orders of $e^2\pi_v^2$, $a_2 f^2\pi_v^2$ can be any even or odd integer greater than or equal to two, we can choose e, f so that $\mathrm{N}_{k_2/k_v}(\eta') - a_2 \equiv 0 \ (\mathfrak{p}_v^{i+1})$. By the first congruence, we still have $\mathrm{Tr}_{k_2/k_v}(\eta') - a_1 \equiv 0 \ (\mathfrak{p}_v^i)$. This proves (3.14.1). If $\mathrm{ord}_{k_v}(\mathrm{Tr}_{k_2/k_v}(\eta) - a_1) = i$, $\mathrm{ord}_{k_v}(\mathrm{Tr}_{k_2/k_v}(\eta') - a_1) = i$ by the first congruence also. So this proves (3.14.2).

Consider the case $2 \leqslant i = \ell = m_v + 1$. We have

$$\begin{aligned}
\mathrm{Tr}_{k_2/k_v}(\eta') - a_1 &\equiv 2e\pi_v + \gamma_1 \pi_v^i \ (\mathfrak{p}_v^{i+1}),\\
\mathrm{N}_{k_2/k_v}(\eta') - a_2 &\equiv e^2\pi_v^2 + a_2 f^2 \pi_v^2 + \gamma_2 \pi_v^i \ (\mathfrak{p}_v^{i+1}).
\end{aligned} \tag{14}$$

As long as $e \in \mathcal{O}_v$, $\mathrm{Tr}_{k_2/k_v}(\eta') - a_1 \equiv 0 \ (\mathfrak{p}_v^i)$. By the same consideration as the previous case, we can choose e, f so that $\mathrm{N}_{k_2/k_v}(\eta') - a_2 \equiv 0 \ (\mathfrak{p}_v^{i+1})$. This proves (3.14.1). We now turn to (3.14.2). By assumption, $\gamma_2 \pi_v^i \equiv \mathrm{N}_{k_2/k_v}(\eta') - a_2 \equiv 0 \ (\mathfrak{p}_v^{i+1})$. So $e^2\pi_v^2 + a_2 f^2 \pi_v^2 \equiv 0 \ (\mathfrak{p}_v^{i+1})$. Since the orders of $e^2\pi_v^2, a_2 f^2 \pi_v^2$ are even and odd, $e^2\pi_v^2, a_2 f^2 \pi_v^2 \equiv 0 \ (\mathfrak{p}_v^{i+1})$. Since $i+1 \geqslant 3$, $e \in \mathfrak{p}_v$. So $2e\pi_v \equiv 0 \ (\mathfrak{p}_v^{i+1})$. This implies that $\mathrm{ord}_{k_v}(\mathrm{Tr}_{k_2/k_v}(\eta') - a_1) = i$ which proves (3.14.2).

We now assume $\ell < i$. Since $i < 2\ell$ by assumption, $\ell > 1$. We first consider the case $\ell \leqslant m_v$. Then $\mathrm{Tr}_{k_2/k_v}(\eta') - a_1 \equiv 0 \ (\mathfrak{p}_v^i)$ if and only if there exists $h \in \mathcal{O}_v$ such that $f = -2e/a_1 + h\pi_v^{i-\ell-1}$. Then by (12),

$$\mathrm{Tr}_{k_2/k_v}(\eta') - a_1 \equiv (a_1/\pi_v^\ell) h \pi_v^i + \gamma_1 \pi_v^i \ (\mathfrak{p}_v^{i+1}),$$

$$\begin{aligned}
\mathrm{N}_{k_2/k_v}(\eta') - a_2 &\equiv e^2\pi_v^2 + a_1 e\pi_v + a_1 e \pi_v^2(-2e/a_1 + h\pi_v^{i-\ell-1})\\
&\quad + \gamma_2 \pi_v^i + 2a_2\pi_v(-2e/a_1 + h\pi_v^{i-\ell-1})\\
&\quad + a_2\pi_v^2(-2e/a_1 + h\pi_v^{i-\ell-1})^2\\
&\equiv (-1 + 4a_2/a_1^2)(e^2\pi_v^2 - a_1 e\pi_v)\\
&\quad + a_2 h^2 \pi_v^{2(i-\ell)} + 2a_2 h \pi_v^{i-\ell} + \gamma_2 \pi_v^i \ (\mathfrak{p}_v^{i+1}).
\end{aligned} \tag{15}$$

Let $N_1 = \mathrm{ord}_{k_v}(e)$, $N_2 = \mathrm{ord}_{k_v}(h)$. Consider (3.14.1). We choose e, h so that $0 \leqslant N_1 < \ell - 1$ and $0 \leqslant N_2 < m_v - i + \ell$. This is possible because $\ell > 1$. Then $\mathrm{ord}_{k_v}(e^2\pi_v^2) < \mathrm{ord}_{k_v}(a_1 e\pi_v)$ and $\mathrm{ord}_{k_v}(h^2\pi_v^{2(i-\ell)}) < \mathrm{ord}_{k_v}(2h\pi_v^{i-\ell})$. Note that $\mathrm{ord}_{k_v}(e^2\pi_v^2) = 2N_1 + 2 < 2\ell$ and it can be any even integer between 2 and $2\ell - 2$. Also $\mathrm{ord}_{k_v}(a_2 h^2 \pi_v^{2(i-\ell)}) < 2m_v + 1$ and it can be any odd integer between $2(i-\ell)+1$ and $2m_v - 1$. Since $i < 2\ell$, $2(i-\ell)+1 \leqslant i$. So we can choose e, h so that $\mathrm{N}_{k_2/k_v}(\eta') - a_2 \equiv 0 \ (\mathfrak{p}_v^{i+1})$. Since $h \in \mathcal{O}_v$, the condition $\mathrm{Tr}_{k_2/k_v}(\eta') - a_1 \equiv 0 \ (\mathfrak{p}_v^i)$ is still satisfied. This proves (3.14.1).

Consider (3.14.2). If $N_2 \geqslant m_v - i + \ell$ then $N_2 > 0$ because $m_v - i + \ell \geqslant 2\ell - i > 0$, by assumption. So $h \in \mathfrak{p}_v$. Therefore, $\mathrm{ord}_{k_v}(\mathrm{Tr}_{k_2/k_v}(\eta') - a_1) = i$. So we assume that $N_2 < m_v - i + \ell$. If $N_1 \geqslant \ell - 1$ then $e^2\pi_v^2 - a_1 e\pi_v \in \mathfrak{p}_v^{2\ell} \subseteq \mathfrak{p}_v^{i+1}$ and so $a_2 h^2 \pi_v^{2(i-\ell)} \in \mathfrak{p}_v^{i+1}$. If $N_1 < \ell - 1$ then $e^2\pi_v^2 + a_2 h^2 \pi_v^{2(i-\ell)} \in \mathfrak{p}_v^{i+1}$. Since the orders of these elements are even and odd, $a_2 h^2 \pi_v^{2(i-\ell)} \in \mathfrak{p}_v^{i+1}$. In both cases, $h^2 \in \mathfrak{p}_v^{i-2(i-\ell)} = \mathfrak{p}_v^{2\ell - i}$. Since $2\ell - i > 0$, $h \in \mathfrak{p}_v$. Therefore, $\mathrm{ord}_{k_v}(\mathrm{Tr}_{k_2/k_v}(\eta') - a_1) = i$. This proves (3.14.2).

We now assume $\ell = m_v + 1$ and so $i \leqslant 2m_v$. Then by (12), $\mathrm{Tr}_{k_2/k_v}(\eta') - a_1 \equiv 0 \ (\mathfrak{p}_v^i)$ if and only if there exists $h \in \mathcal{O}_v$ such that $e = -(a_1/2)f + h\pi_v^{i-m_v-1}$. Then

$$\mathrm{Tr}_{k_2/k_v}(\eta') - a_1 \equiv (2/\pi_v^{m_v})h\pi_v^i + \gamma_1\pi_v^i \ (\mathfrak{p}_v^{i+1}),$$

$$\begin{aligned}\mathrm{N}_{k_2/k_v}(\eta') - a_2 &\equiv ((-a_1/2)f + h\pi_v^{i-m_v-1})^2\pi_v^2 \\ &\quad + a_1\pi_v(1+f\pi_v)((-a_1/2)f + h\pi_v^{i-m_v-1}) \\ &\quad + 2a_2f\pi_v + a_2f^2\pi_v^2 + \gamma_2\pi_v^i \\ &\equiv (a_2 - (a_1^2/4))f^2\pi_v^2 + (2a_2 - (a_1^2/2))f\pi_v \\ &\quad + h^2\pi_v^{2(i-m_v)} + \gamma_2\pi_v^i \ (\mathfrak{p}_v^{i+1}). \end{aligned} \tag{16}$$

Let $a_2 - (a_1^2/4) = r\pi_v$ and $2a_2 - (a_1^2/2) = s\pi_v^{m_v+1}$. Then it is easy to see that $r, s \in \mathcal{O}_v^\times$.

Suppose $N = \mathrm{ord}_{k_v}(f)$. Consider (3.14.1). We choose $0 \leqslant N < m_v - 1$. Then $\mathrm{ord}_{k_v}(rf^2\pi_v^3) = 2N + 3 < N + m_v + 2 = \mathrm{ord}_{k_v}(sf\pi_v^{m_v+2})$ and $\mathrm{ord}_{k_v}(rf^2\pi_v^3) = 2N + 3$ can be any odd integer between 3 and $2m_v - 1$. Since $i \leqslant 2m_v$, $2(i - m_v) - i = i - 2m_v \leqslant 0$. So $\mathrm{ord}_{k_v}(h^2\pi_v^{2(i-m_v)})$ can be any even integer greater than or equal to i. Therefore, we can choose f, h so that $\mathrm{N}_{k_2/k_v}(\eta') - a_2 \equiv 0 \ (\mathfrak{p}_v^{i+1})$. This proves (3.14.1).

Consider (3.14.2). By assumption,

$$rf^2\pi_v^3 + sf\pi_v^{m_v+2} + h^2\pi_v^{2(i-m_v)} \equiv 0 \ (\mathfrak{p}_v^{i+1}).$$

If $N \geqslant m_v - 1$ then

$$rf^2\pi_v^3 + rf\pi_v^{m_v+2} \in \mathfrak{p}_v^{2m_v+1} \subseteq \mathfrak{p}_v^{i+1}.$$

So $h^2\pi_v^{2(i-m_v)} \equiv 0 \ (\mathfrak{p}_v^{i+1})$. If $N < m_v - 1$ then it follows that we have $\mathrm{ord}_{k_v}(rf^2\pi_v^3) < \mathrm{ord}_{k_v}(rf\pi_v^{m_v+2})$ and the orders of $rf^2\pi_v^3$ and $h^2\pi_v^{2(i-m_v)}$ are odd and even, respectively. This implies that $h^2\pi_v^{2(i-m_v)} \equiv 0 \ (\mathfrak{p}_v^{i+1})$ also. In both cases, $h^2 \in \mathfrak{p}_v^{2m_v+1-i} \subseteq \mathfrak{p}_v$ and so $h \in \mathfrak{p}_v$. Therefore, $\mathrm{ord}_{k_v}(\mathrm{Tr}_{k_2/k_v}(\eta') - a_1) = i$. This proves (3.14.2). $\square$

The following corollary is easily deduced from the proposition.

Corollary 3.15. *The level of k_1, k_2 is i if and only if there exists $\eta \in \mathcal{O}_2$ such that*

$$\mathrm{ord}_{k_v}(\mathrm{Tr}_{k_2/k_v}(\eta) - a_1) = i, \quad \mathrm{N}_{k_2/k_v}(\eta) \equiv a_2 \ (\mathfrak{p}_v^{i+1}).$$

As we discussed in Section 7 of [3], the following proposition provides a relation between the level and the relative discriminant of $k_1 \cdot k_2/k_2$.

Proposition 3.16. (3.16.1) *Suppose* $\ell_1 = \ell_2 = \ell$, $\delta_1 = \delta_2 = \delta$ *and* $\ell \leqslant \mathrm{lev}(k_1, k_2) \leqslant \delta$. *Let* $i = \mathrm{lev}(k_1, k_2)$. *Then we have* $\Delta_{k_1 \cdot k_2/k_2} = \mathfrak{p}_2^{2(\delta - i)}$ *and* $\Delta_{k_3/k_v} = \mathfrak{p}_v^{2(\delta - i)}$.

(3.16.2) *Suppose* $\ell_1 > \ell_2$. *Then we have* $\Delta_{k_1 \cdot k_2/k_1} = \mathfrak{p}_1^{\delta_2}$, $\Delta_{k_1 \cdot k_2/k_2} = \mathfrak{p}_2^{2\delta_1 - \delta_2}$.

Proof. Consider the first claim in (3.16.1). If $\mathrm{lev}(k_1, k_2) = \delta$ then $k_1 \cdot k_2/k_2$ and k_3/k_v are unramified, by Proposition 3.10, and so (3.16.1) holds. We may now assume that $i < \delta$. We choose η which satisfies the condition of Corollary 3.15. Let $p(z) = z^2 + a_1 z + a_2 = 0$ be the Eisenstein equation with roots $\alpha = \{\alpha_1, \alpha_2\}$, which generate the field k_1. Let $\gamma = \pi_2^{-i}(\alpha_1 + \eta)$. Then $p(\alpha_1) = 0$ is equivalent to the following equation:

$$\gamma^2 + \pi_2^{-i}(a_1 - 2\eta)\gamma + \pi_2^{-2i}(\eta^2 - a_1\eta + a_2) = 0. \tag{17}$$

Since $\eta^2 - a_1\eta + a_2 = \eta(\mathrm{Tr}_{k_2/k_v}(\eta) - a_1) + a_2 - \mathrm{N}_{k_2/k_v}(\eta)$, the order of the third term in (17) is one. Therefore, (17) is an Eisenstein equation whose roots generate $k_1 \cdot k_2/k_2$. If $\ell \leqslant m_v$ then $\mathrm{ord}_{k_2}(a_1) = 2\ell$ and $\mathrm{ord}_{k_2}(2\eta) = 2m_v + 1 > 2\ell$. So $\mathrm{ord}_{k_2}(\pi_2^{-i}(a_1 - 2\eta)) = \delta - i$. If $\ell = m_v + 1$ then $\mathrm{ord}_{k_2}(a_1) = 2m_v + 2$ and $\mathrm{ord}_{k_2}(2\eta) = 2m_v + 1 = \delta$. So $\mathrm{ord}_{k_2}(\pi_2^{-i}(a_1 - 2\eta)) = \delta - i$ also. Since $2\mathcal{O}_2 = \mathfrak{p}_2^{2m_v}$ and $\delta - i \leqslant 2m_v$, $\Delta_{k_1 \cdot k_2/k_2} = \mathfrak{p}_2^{2(\delta - i)}$.

Consider the second claim in (3.16.1). By Corollary 3.15, we choose an element $\eta \in S_{i,i+1}(k_1, k_2)$. We may assume that $-\eta$ is one of the roots of $p_2(z)$. Let $p_3(x)$ be polynomial (9). Then the roots of $p_3(z)$ generate the field k_3. We evaluate the order of the element J in Lemma 3.5, which is the same as that in (10).

By assumption $\mathrm{ord}_{k_v}(a_2 - b_2) \geqslant i + 1$, $\mathrm{ord}_{k_v}(a_1 - b_1) = i$, $\mathrm{ord}_{k_v}(a_1(b_2 - a_2)) \geqslant i + 2$ and $\mathrm{ord}_{k_v}(a_2(a_1 - b_1)) = i + 1$. Therefore, $\mathrm{ord}_{k_v}(J) = 2i + 1$. Now

$$\pi_v^{-2i} p_3(\pi_v^i z) = z^2 - \pi_v^{-i}[2(a_2 + b_2) - a_1 b_1]z + \pi_v^{-2i} J. \tag{18}$$

Note that $2(a_2 + b_2) = 4a_2 + 2(b_2 - a_2)$, $\mathrm{ord}_{k_v}(4a_2) = 2m_v + 1$ and also that $\mathrm{ord}_{k_v}(2(b_2 - a_2)) \geqslant m_v + i + 1$. If $\ell \leqslant m_v$ then $\mathrm{ord}_{k_v}(a_1 b_1) = 2\ell < 2m_v + 1, m_v + i + 1$ since $i \geqslant \ell$. This implies that the order of $\pi_v^{-i}[2(a_2 + b_2) - a_1 b_1]$ is $2\ell - i = \delta - i$. If $\ell = m_v + 1$ then $\mathrm{ord}_{k_v}(a_1 b_1) \geqslant 2m_v + 2 = \delta + 1$. Since $i \geqslant m_v + 1$, the order of $2\pi_v^{-i}(a_2 + b_2)$ is $2m_v + 1 - i = \delta - i$. Therefore, in both cases, (18) is an Eisenstein polynomial with the order of the coefficient of the middle term $\delta - i \leqslant m_v$. Therefore, $\Delta_{k_3/k_v} = \mathfrak{p}_v^{2(\delta - i)}$.

Consider (3.16.2). By Lemma 3.6, $\delta_3 = \delta_1 > \delta_2$. Let $i = \mathrm{lev}(k_1, k_3)$. Then $\delta_2 = 2(\delta_1 - i)$ by the second statement of (3.16.1). Therefore, using the first statement of (3.16.1), $\Delta_{k_1 \cdot k_2/k_v} = \Delta_{k_1 \cdot k_3/k_v} = \mathfrak{p}_v^{2\delta_1 + \delta_2}$ (see [4, Corollary 4, p. 142] which is a local version of [4], Proposition 13, p. 156). This implies that $\Delta_{k_1 \cdot k_2/k_1} = \mathfrak{p}_1^{\delta_2}$ and $\Delta_{k_1 \cdot k_2/k_2} = \mathfrak{p}_2^{2\delta_1 - \delta_2}$. Thus (3.16.2). $\square$

We now review the equivalence relation $x \asymp y$ and explain the notation in the introduction. Since we are only concerned with x such that $k_v(x)/k_v$ is ramified, we restrict ourselves to such orbits. Suppose $x, y \in V^{\mathrm{ss}}_{k_v}$ and $k_v(x), k_v(y)$ are ramified quadratic extensions of k_v. If the type of x is (rm rm)* or (rm rm ur), $x \asymp y$ means x, y are in the same G_{k_v}-orbit. If the type of x is (rm rm rm), we write $x \asymp y$ if and only if $\Delta_{k_v(x)/k_v} = \Delta_{k_v(y)/k_v}$ and $\mathrm{lev}(k_v(x), \tilde{k_v}) = \mathrm{lev}(k_v(y), \tilde{k_v})$. By Proposition 3.16, the last condition is equivalent to the condition $\Delta_{\tilde{k_v}(x)/\tilde{k_v}} = \Delta_{\tilde{k_v}(y)/\tilde{k_v}}$. If $k_v(x)/k_v$ is ramified, we let $\Delta_{k_v(x)/k_v} = \mathfrak{p}_v^{\delta_{x,v}}$ and $\lambda_{x,v} = \mathrm{lev}(k_v(x), \tilde{k_v})$. This explains the notation in Tables 1 and 2.

As we promised earlier, we explain the motivation for our formulation. Before finally choosing the formulation of the filtering process in Section 6 of [3], we carried out some experiments. At first we tried to compute the standard local zeta functions explicitly and we did succeed for non-dyadic places, even though we later settled on a uniform estimate without the explicit forms to shorten the paper. Then we worked on dyadic places and we discovered that it is difficult even to determine the constant terms of the standard local zeta functions. If one tries to compute them, the set $S_{i_1,i_2}(k_1, k_2)$ naturally occurs. In fact, if v is dyadic and an orbit of $x \in V^{\mathrm{ss}}_{k_v}$ corresponds to a field $k_v(x)$ such that $\mathrm{lev}(k_v(x), \tilde{k_v}) = i$, it turns out that the constant term of the standard local zeta function is $\sum_{j=0}^{i} (\mathfrak{n}_1(k_v(x), \tilde{k_v}, j) + \mathfrak{n}_2(k_v(x), \tilde{k_v}, j))$. We also evaluated the terms in this sum and the answer was that if x is of type (rm rm rm) then $\mathfrak{n}_r(k_v(x), \tilde{k_v}, j) = q_v^j$ for $r = 1, 2$ and $j \leqslant i$ and if x is of type (rm rm ur) then $\mathfrak{n}_r(k_v(x), \tilde{k_v}, j) = q_v^j$ for $r = 1, 2$ and $j \leqslant \tilde{\delta} - 1$ and $\mathfrak{n}_1(k_v(x), \tilde{k_v}, \tilde{\delta}) = q_v^{\tilde{\delta}}, \mathfrak{n}_2(k_v(x), \ \tilde{k_v}, \tilde{\delta}) = 0$. In the process we had to prove something like Proposition 3.14. We realized later that we did not need the constant term nor any estimate of the standard local zeta functions at dyadic places, but having Proposition 3.14 eventually helped us to evaluate the local densities at dyadic places. This was our motivation for introducing the set $S_{i_1,i_2}(k_1, k_2)$.

4. The volume of the integral points of the stabilizer

In this section we evaluate $\mathrm{vol}(K_v \cap G^{\circ}_{x\,k_v})$ for orbits of types (rm rm ur) and (rm rm rm). The measure on $G^{\circ}_{x\,k_v}$ is defined in Definition 5.13 of [3]. We shall not repeat the definition here but instead recall an alternative formula. Consider the usual multiplicative measure on $\tilde{k_v}(x)$, i.e. that for which the volume of $\mathcal{O}^{\times}_{\tilde{k_v}(x)}$ is 1. Suppose α_1 is a uniformizer of $k_v(x)$. Then it was proved in Lemma 3.4 of [2] that

$$\mathrm{vol}(K_v \cap G^{\circ}_{x\,k_v}) = \mathrm{vol}(\tilde{\mathcal{O}}_v[\alpha_1]^{\times}). \tag{19}$$

To determine the volume in the above cases we need the following result, which provides a reinterpretation of the level in these cases.

Proposition 4.1. *Suppose that $k_1, k_2/k_v$ are distinct ramified quadratic extensions. Let $\mathcal{O}_i$ be the integer ring of k_i, $\mathfrak{p}_i$ be the prime ideal of $\mathcal{O}_i$ and π_i be its uniformizer for $i = 1, 2$. We denote the integer ring of $k_1 \cdot k_2$ by $\mathcal{O}_{k_1 \cdot k_2}$. Let $p_1(z) = z^2 + a_1 z + a_2$ be an*

Eisenstein polynomial defining k_1 *and* $\alpha_1 \in \mathcal{O}_1$ *be a root of* p_1. *Let* f *be the least integer such that* $\mathfrak{p}_2^f \cdot \mathcal{O}_{k_1 \cdot k_2} \subseteq \mathcal{O}_2[\alpha_1]$. *Then* $f = \mathrm{lev}(k_1, k_2)$.

Proof. We shall show first that $f \geqslant \mathrm{lev}(k_1, k_2)$ in general. Let $i = \mathrm{lev}(k_1, k_2)$ and choose $\beta_1 \in \mathcal{O}_2$ with minimal polynomial $z^2 + b_1 z + b_2$ such that $a_1 \equiv b_1\ (\mathfrak{p}_v^i)$ and $a_2 \equiv b_2\ (\mathfrak{p}_v^i)$. Consider the element $(\alpha_1 - \beta_1)/\pi_2^i$ of $k_1 \cdot k_2$; we claim that it is an integer. In fact, by Lemma 3.5,

$$
\begin{aligned}
&\mathrm{N}_{k_1 \cdot k_2/k_v}((\alpha_1 - \beta_1)/\pi_2^i) \\
&\quad = \frac{1}{\mathrm{N}_{k_2/k_v}(\pi_2)^{2i}} [(a_2 - b_2)^2 + (a_1 - b_1)(a_1 b_2 - a_2 b_1)] \\
&\quad = \frac{1}{\mathrm{N}_{k_2/k_v}(\pi_2)^{2i}} [(a_2 - b_2)^2 + (a_1 - b_1)(a_1(b_2 - a_2) + a_2(a_1 - b_1))] \in \mathcal{O}_v
\end{aligned}
$$

by hypothesis. Thus $\pi_2^{f-i}(\alpha_1 - \beta_1) \in \mathcal{O}_2[\alpha_1]$ and it follows that $f \geqslant i$, as claimed.

We know that $\mathrm{lev}(k_1, k_2) \geqslant 1$ and so $f \geqslant 1$ and $\mathrm{lev}(k_1, k_2) = f$ if $f = 1$. We now assume that $f \geqslant 2$ to complete the proof. There are $\eta, \zeta \in \mathcal{O}_2$ such that $(\eta + \zeta\alpha_1)/\pi_2^f \in \mathcal{O}_{k_1 \cdot k_2}$ and one of η, ζ is a unit (for otherwise f would not be the least integer with its defining property). Taking norms from $k_1 \cdot k_2$ to k_2 we find that $\eta^2 - a_1\eta\zeta + a_2\zeta^2 \in \mathfrak{p}_2^{2f}$. Since $a_1, a_2 \in \mathfrak{p}_v \subseteq \mathfrak{p}_2^2$, $\eta \in \mathfrak{p}_2$. It follows that $\zeta \in \mathcal{O}_2^\times$. Furthermore, $\eta^2 + a_2\zeta^2 \in \mathfrak{p}_2^3$ and $a_2\zeta^2 \in \mathfrak{p}_2^2 \backslash \mathfrak{p}_2^3$ from which it follows that $\eta \in \mathfrak{p}_2 \backslash \mathfrak{p}_2^2$. Let us set $\varpi = \eta/\zeta$. Then ϖ is a uniformizer of k_2 and $\varpi^2 - a_1\varpi + a_2 \in \mathfrak{p}_2^{2f}$. Let $z^2 - c_1 z + c_2$ be the (Eisenstein) minimal polynomial of ϖ over k_v. Then

$$
\begin{aligned}
(c_1 - a_1)\varpi + (a_2 - c_2) &= (\varpi^2 - a_1\varpi + a_2) - (\varpi^2 - c_1\varpi + c_2) \\
&= (\varpi^2 - a_1\varpi + a_2) \in \mathfrak{p}_2^{2f}.
\end{aligned}
$$

Since $(c_1 - a_1)\varpi$ has odd order in k_2 and $(a_2 - c_2)$ has even order, it follows that $(a_2 - c_2) \in \mathfrak{p}_2^{2f} \cap \mathcal{O}_v = \mathfrak{p}_v^f$ and $(c_1 - a_1)\varpi \in \mathfrak{p}_2^{2f+1}$ which implies that $(c_1 - a_1) \in \mathfrak{p}_v^f$. Thus $f \leqslant \mathrm{lev}(k_1, k_2)$. This proves the proposition. $\square$

Proposition 4.2. *If* x *is the standard orbital representative for an orbit with type* (rm rm ur) *then* $\mathrm{vol}(K_v \cap G^\circ_{x\,k_v}) = (1 + q_v^{-1})^{-1} q_v^{-\tilde{\delta}_v}$. *If* x *is the standard orbital representative for an orbit with type* (rm rm rm) *then* $\mathrm{vol}(K_v \cap G^\circ_{x\,k_v}) = q_v^{-i}$ *where* $i = \mathrm{lev}(k_v(x), \tilde{k}_v)$.

Proof. The ring $\tilde{\mathcal{O}}_v[\alpha_1]$ is an $\tilde{\mathcal{O}}_v$-order in $\mathcal{O}_{\tilde{k}_v(x)}$ and so if $\beta_1 \in \mathcal{O}_{\tilde{k}_v(x)}$ satisfies $\mathcal{O}_{\tilde{k}_v(x)} = \tilde{\mathcal{O}}_v[\beta_1]$ then there is some $i \geqslant 0$ such that

$$
\tilde{\mathcal{O}}_v[\alpha_1] = \{a + b\beta_1 \mid a \in \tilde{\mathcal{O}}_v, b \in \tilde{\mathfrak{p}}_v^i\}.
$$

From the previous proposition we see that $i = \operatorname{lev}(k_v(x), \tilde{k}_v) \geqslant 1$. Then

$$\tilde{\mathcal{O}}_v[\alpha_1]^\times = \{a + b\beta_1 \mid a \in \tilde{\mathcal{O}}_v^\times, b \in \tilde{\mathfrak{p}}_v^i\}.$$

The normalized additive Haar measure on $\mathcal{O}_{\tilde{k}_v(x)}$ is $da\,db$ and so the normalized multiplicative Haar measure on $\mathcal{O}_{\tilde{k}_v(x)}^\times$ is $(1 - q_{\tilde{k}_v(x)}^{-1})^{-1} da\,db$, where $q_{\tilde{k}_v(x)}$ is the module of $\tilde{k}_v(x)$. Since $\tilde{\mathcal{O}}_v/\tilde{\mathfrak{p}}_v \cong \mathcal{O}_v/\mathfrak{p}_v$,

$$\operatorname{vol}(\tilde{\mathcal{O}}_v[\alpha_1]^\times) = (1 - q_{\tilde{k}_v(x)}^{-1})^{-1}(1 - q_v^{-1})q_v^{-i}.$$

In case the index is (rm rm ur), $q_{\tilde{k}_v(x)} = q_v^2$ and $i = \tilde{\delta}_v$ and we have $\operatorname{vol}(\tilde{\mathcal{O}}_v[\alpha_1]^\times) = (1 + q_v^{-1})^{-1} q_v^{-\tilde{\delta}_v}$. In case the index is (rm rm rm), $q_{\tilde{k}_v(x)} = q_v$ and we have $\operatorname{vol}(\tilde{\mathcal{O}}_v[\alpha_1]^\times) = q_v^{-i}$. $\square$

5. Orbital volumes at the ramified dyadic places

In this section, we group orbits according to the level and compute $\sum_x \operatorname{vol}(K_v x)$ for each group of orbits.

Let $\tilde{p}(z) = z^2 + b_1 z + b_2$ be an Eisenstein polynomial whose roots $\eta = \{\eta_1, \eta_2\}$ generate $\tilde{k}_v$. Let $\tilde{\ell} = \operatorname{ord}_{k_v}(b_1)$ if $\operatorname{ord}_{k_v}(b_1) \leqslant m_v$ and $\tilde{\ell} = m_v + 1$ if $\operatorname{ord}_{k_v}(b_1) \geqslant m_v + 1$. In the first case let $\tilde{\delta}_v = 2\tilde{\ell}$ and in the second case let $\tilde{\delta}_v = 2m_v + 1$, as before. For an Eisenstein polynomial $p_1(z) = z^2 + a_1 z + a_2 = 0$ define $\ell(p_1)$ and $\delta(p_1)$ similarly.

Definition 5.1. (5.1.1) If $\ell_1 \neq \tilde{\ell}$ then $\mathfrak{X}_{\ell_1}$ is the set of isomorphism classes of quadratic extensions k' of k_v generated by roots of an Eisenstein equation $p_1(z) = z^2 + a_1 z + a_2 = 0$ such that $\ell(p_1) = \ell_1$.

(5.1.2) If $\tilde{\ell} \leqslant i < \tilde{\delta}_v$ then $\mathfrak{X}_{\tilde{\ell}}(i)$ is the set of isomorphism classes of quadratic extensions k' of k_v generated by roots of an Eisenstein equation $p_1(z) = z^2 + a_1 z + a_2 = 0$ such that $\ell(p_1) = \tilde{\ell}$ and $\operatorname{lev}(k', \tilde{k}_v) = i$.

(5.1.3) We define $\mathfrak{X}_{\tilde{\ell}}^{\mathrm{ur}}$ to be the singleton containing the unique quadratic extension of k_v of type (rm rm ur).

(5.1.4) We define $\mathfrak{X}_{\tilde{\ell}}^{*}$ to be the singleton containing the unique quadratic extension of k_v of type (rm rm)*.

Let $a_0(x), a_1(x), a_2(x)$ be as in (5) of Section 2. For each type of x in Definition 5.1, we compute $\sum_x \operatorname{vol}(K_v x)$. Our strategy is the same as that in Section 4 of [2]; we define a subset $\mathcal{D} \subseteq V_{\mathcal{O}_v}$ using congruence conditions, cover $K_v x$ by disjoint copies of $\mathcal{D}$ and count the number of copies. Our first task is to define the set $\mathcal{D}$ for each case, which we shall do as follows.

Let $\ell_1 \neq \tilde{\ell}$. We put $\ell = \min\{\ell_1, \tilde{\ell}\}$. We define $\mathscr{D}_{\ell_1}$ to be the set of x which satisfy the conditions

$$\begin{aligned} & x_{11} \in \tilde{\mathcal{O}}_v^\times, \quad x_{20} \in \mathcal{O}_v^\times, \\ & \mathrm{ord}_{\tilde{k_v}}(x_{21}) = 1, \quad \mathrm{ord}_{k_v}(x_{12}) = \ell, \quad \mathrm{ord}_{k_v}(x_{22}) \geqslant \ell + 1, \\ & \mathrm{ord}_{k_v}(a_1(x)) \begin{cases} = \ell_1 & \text{if } \ell_1 \leqslant m_v, \\ \geqslant m_v + 1 & \text{if } \ell_1 = m_v + 1. \end{cases} \end{aligned} \tag{20}$$

We define $\mathscr{D}_{\tilde{\ell}}(i)$ to be the set of x which satisfy the conditions

$$\begin{aligned} & x_{11} \in \tilde{\mathcal{O}}_v^\times, \quad x_{20} \in \mathcal{O}_v^\times, \\ & \mathrm{ord}_{\tilde{k_v}}(x_{21}) = 1, \quad \mathrm{ord}_{k_v}(x_{12}) = i, \quad \mathrm{ord}_{k_v}(x_{22}) \geqslant i + 1, \\ & \mathrm{ord}_{k_v}(a_1(x)) \begin{cases} = \tilde{\ell} & \text{if } \tilde{\ell} \leqslant m_v, \\ \geqslant m_v + 1 & \text{if } \tilde{\ell} = m_v + 1. \end{cases} \end{aligned} \tag{21}$$

We define $\mathscr{D}_{\tilde{\ell}}^{\mathrm{ur}\,*}$ to be the set of x which satisfy the conditions

$$\begin{aligned} & x_{11} \in \tilde{\mathcal{O}}_v^\times, \ x_{20} \in \mathcal{O}_v^\times, \\ & \mathrm{ord}_{\tilde{k_v}}(x_{21}) = 1, \quad \mathrm{ord}_{k_v}(x_{12}), \quad \mathrm{ord}_{k_v}(x_{22}) \geqslant \tilde{\delta}_v, \\ & \mathrm{ord}_{k_v}(a_1(x)) \begin{cases} = \tilde{\ell} & \text{if } \tilde{\ell} \leqslant m_v, \\ \geqslant m_v + 1 & \text{if } \tilde{\ell} = m_v + 1. \end{cases} \end{aligned} \tag{22}$$

Let $\eta = (\eta_1, \eta_2), \tilde{p}(z)$ be as in the beginning of this section. We define

$$w_\eta = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -\eta_2 \\ -\eta_1 & 0 \end{pmatrix} \right). \tag{23}$$

Then $k_v(w_\eta) = \tilde{k_v}$. Note that $w_\eta = (n(\eta_2), 1) w_{\tilde{p}}$. We define

$$\mathscr{D}_{\tilde{\ell}}^{*} = \{ x \in V_{\mathcal{O}_v} \mid x \equiv w_\eta(\mathfrak{p}_v^{\tilde{\delta}_v + 1}, \tilde{\mathfrak{p}}_v^{2(\tilde{\delta}_v + 1)}) \}. \tag{24}$$

Our next task is to show that points in the above sets correspond to fields of types (5.1.1)–(5.1.4) in Definition 5.1. Given points in the above sets, we try to simplify them by group elements as much as possible so that, after the simplification, the types of the corresponding fields are easy to determine. For this purpose we define subgroups of K_v which stabilize the above sets. They will also be used later for the computation of $\sum_x \mathrm{vol}(K_v x)$. We use the coordinate system (3) of Section 2.

Definition 5.2. For $j \geqslant 0$ we define

$$H(j) = \{g = (g_1, g_2) \in G_{\mathcal{O}_v} \mid g_{121} \in \tilde{\mathfrak{p}}_v^{j+1}, g_{221} \in \mathfrak{p}_v\},$$

$$\bar{H}(j) = \{g = (g_1, g_2) \in \mathrm{GL}(2)_{\tilde{\mathcal{O}}_v/\tilde{\mathfrak{p}}_v^{j+1}} \times \mathrm{GL}(2)_{\mathcal{O}_v/\mathfrak{p}_v} \mid g_{121} = 0, g_{221} = 0\},$$

$$G(\pi_v^{\tilde{\delta}_v+1}) = \{g = (g_1, g_2) \in G_{\mathcal{O}_v} \mid g_1 \equiv 1 \ (\tilde{\mathfrak{p}}_v^{2(\tilde{\delta}_v+1)}), g_2 \equiv 1 \ (\mathfrak{p}_v^{\tilde{\delta}_v+1})\}.$$

We put

$$\begin{aligned} Q(j) &= \#(K_v/H(j)) = \#(\mathrm{GL}(2)_{\tilde{\mathcal{O}}_v/\tilde{\mathfrak{p}}_v^{j+1}} \times \mathrm{GL}(2)_{\mathcal{O}_v/\mathfrak{p}_v})/\#\bar{H}(j) \\ &= \frac{(q_v^2 - q_v)^2 (q_v^2 - 1)^2 q_v^{4j}}{(q_v - 1)^2 q_v^{2j} q_v^{j+1} (q_v - 1)^2 q_v} \\ &= q_v^{j+2}(1 + q_v^{-1})^2. \end{aligned} \tag{25}$$

Proposition 5.3.

(5.3.1) *If* $\ell_1 \neq \tilde{\ell}$ *and* $\ell = \min\{\ell_1, \tilde{\ell}\}$ *then* $H(\ell)\mathcal{D}_{\ell_1} = \mathcal{D}_{\ell_1}$.
(5.3.2) *If* $\tilde{\ell} \leqslant i < \tilde{\delta}_v$ *then* $H(i)\mathcal{D}_{\tilde{\ell}}(i) = \mathcal{D}_{\tilde{\ell}}(i)$.
(5.3.3) *We have* $H(\tilde{\delta}_v - 1)\mathcal{D}_{\tilde{\ell}}^{\mathrm{ur}\,*} = \mathcal{D}_{\tilde{\ell}}^{\mathrm{ur}\,*}$.
(5.3.4) *We have* $G(\pi_v^{\tilde{\delta}_v+1})\mathcal{D}_{\tilde{\ell}}^{*} = \mathcal{D}_{\tilde{\ell}}^{*}$.

Proof. Part (5.3.4) is obvious. Consider (5.3.1)–(5.3.3). We put $j = \ell, i, \tilde{\delta}_v - 1$ for (5.3.1)–(5.3.3), respectively. Then the group in question is $H(j)$ in all parts.

If $g = (g_1, g_2) \in H(j)$ then $(g_1, 1), (1, g_2) \in H(j)$ and $g = (1, g_2)(g_1, 1)$. Thus it is enough to verify the claims for $g = (g_1, 1)$ and $g = (1, g_2)$ separately. We begin with $g = (g_1, 1)$. For x in the form (4) of Section 2, let $y = gx = (y_1, y_2)$ and consider similar coordinates for y. Then

$$\begin{aligned} y_{r0} &= \mathrm{N}_{\tilde{k}_v/k_v}(g_{111})x_{r0} + \mathrm{Tr}_{\tilde{k}_v/k_v}(g_{111}g_{112}^{\sigma}x_{r1}) + \mathrm{N}_{\tilde{k}_v/k_v}(g_{112})x_{r2}, \\ y_{r1} &= g_{111}g_{121}^{\sigma}x_{r0} + g_{111}g_{122}^{\sigma}x_{r1} + g_{112}g_{121}^{\sigma}x_{r1}^{\sigma} + g_{112}g_{122}^{\sigma}x_{r2}, \\ y_{r2} &= \mathrm{N}_{\tilde{k}_v/k_v}(g_{121})x_{r0} + \mathrm{Tr}_{\tilde{k}_v/k_v}(g_{121}g_{122}^{\sigma}x_{r1}) + \mathrm{N}_{\tilde{k}_v/k_v}(g_{122})x_{r2} \end{aligned} \tag{26}$$

for $r = 1, 2$.

Suppose $x \in \mathcal{D}_{\ell_1}, \mathcal{D}_{\tilde{\ell}}(i)$ or $\mathcal{D}_{\tilde{\ell}}^{\mathrm{ur}\,*}$. Note that $j + 1 \geqslant 2$ in all cases. So $g_{121}, g_{121}^{\sigma} \in \tilde{\mathfrak{p}}_v^{j+1} \subseteq \tilde{\mathfrak{p}}_v^2$, $x_{12} \in \mathfrak{p}_v$ and $x_{11} \in \tilde{\mathcal{O}}_v^{\times}$. Therefore, $y_{11} \in \tilde{\mathcal{O}}_v^{\times}$ by (26). We also have

$y_{20} \in \mathcal{O}_v^\times$ since $\mathrm{N}_{\tilde{k}_v/k_v}(g_{111})x_{20} \in \mathcal{O}_v^\times$, $x_{21} \in \tilde{\mathfrak{p}}_v$ and $x_{22} \in \mathfrak{p}_v$. Since $g_{121}, g_{121}^\sigma \in \tilde{\mathfrak{p}}_v^2$, $x_{22} \in \mathfrak{p}_v \subseteq \tilde{\mathfrak{p}}_v^2$ and $\mathrm{ord}_{\tilde{k}_v}(x_{21}) = 1$, we further have $\mathrm{ord}_{\tilde{k}_v}(y_{21}) = 1$.

Note that $j + 1 \leqslant \tilde{\delta}_v$ in all cases. By Lemma 3.2,

$$\mathrm{ord}_{k_v}(\mathrm{Tr}_{\tilde{k}_v/k_v}(g_{121}g_{122}^\sigma x_{r1})) \geqslant \lfloor (j + 1 + \tilde{\delta}_v)/2 \rfloor \geqslant j + 1.$$

Therefore, $\mathrm{Tr}_{\tilde{k}_v/k_v}(g_{121}g_{122}^\sigma x_{r1}) \in \mathfrak{p}_v^{j+1}$ for $r = 1, 2$. Also $\mathrm{N}_{\tilde{k}_v/k_v}(g_{121}) \in \mathfrak{p}_v^{j+1}$. By assumption $\mathrm{ord}_{k_v}(\mathrm{N}_{\tilde{k}_v/k_v}(g_{122})x_{12}) = j$ in cases (5.3.1), (5.3.2) and so $\mathrm{ord}_{k_v}(y_{12}) = j$. In case (5.3.3), $\mathrm{N}_{\tilde{k}_v/k_v}(g_{122})x_{12} \in \mathfrak{p}_v^{j+1}$ and so $y_{12} \in \mathfrak{p}_v^{j+1}$. In all cases $\mathrm{N}_{\tilde{k}_v/k_v}(g_{122})x_{22} \in \mathfrak{p}_v^{j+1}$ and so $y_{22} \in \mathfrak{p}_v^{j+1}$. We have $a_1(y) = \mathrm{N}_{\tilde{k}_v/k_v}(\det(g_1))a_1(x)$ and $\det(g_1) \in \tilde{\mathcal{O}}_v^\times$. Thus $\mathrm{ord}_{k_v}(a_1(y)) = \mathrm{ord}_{k_v}(a_1(x))$. All the conditions for y to lie in $\mathcal{D}_{\ell_1}$, $\mathcal{D}_{\tilde{\ell}}(i)$ or $\mathcal{D}_{\tilde{\ell}}^{\mathrm{ur}*}$ have now been verified and so $gx \in \mathcal{D}_{\ell_1}, \mathcal{D}_{\tilde{\ell}}(i)$ or $\mathcal{D}_{\tilde{\ell}}^{\mathrm{ur}*}$.

We now assume that $g = (1, g_2)$. It is easy to verify that g preserves all the conditions in (20)–(22), with the possible exception of the last. The necessary calculation to show that this condition is also preserved by g has already been carried out in [2], Lemma 4.24 and will not be repeated here (note that g_3 stands in for g_2 in the proof of Lemma 4.24). This proves the proposition. □

Proposition 5.4. (5.4.1) *If $\ell_1 \neq \tilde{\ell}$ and $x \in \mathcal{D}_{\ell_1}$ then $k_v(x) \in \mathfrak{X}_{\ell_1}$. If $k' \in \mathfrak{X}_{\ell_1}$ then there is some $x \in \mathcal{D}_{\ell_1}$ with $k_v(x) \cong k'$ over k_v.*

(5.4.2) *If $\tilde{\ell} \leqslant i < \tilde{\delta}_v$ and $x \in \mathcal{D}_{\tilde{\ell}}(i)$ then $k_v(x) \in \mathfrak{X}_{\tilde{\ell}}(i)$. If $k' \in \mathfrak{X}_{\tilde{\ell}}(i)$ then there is some $x \in \mathcal{D}_{\tilde{\ell}}(i)$ with $k_v(x) \cong k'$ over k_v.*

(5.4.3) *If $x \in \mathcal{D}_{\tilde{\ell}}^{\mathrm{ur}*}$ then $k_v(x) \in \mathfrak{X}_{\tilde{\ell}}^{\mathrm{ur}} \cup \mathfrak{X}_{\tilde{\ell}}^*$. If $k' \in \mathfrak{X}_{\tilde{\ell}}^{\mathrm{ur}} \cup \mathfrak{X}_{\tilde{\ell}}^*$ then there is some $x \in \mathcal{D}_{\tilde{\ell}}^{\mathrm{ur}*}$ such that $k_v(x) \cong k'$ over k_v.*

(5.4.4) *If $x \in \mathcal{D}_{\tilde{\ell}}^*$ then $k_v(x) = \tilde{k}_v$.*

Proof. We first consider the first implication in each of (1)–(4). Let $x \in \mathcal{D}_{\ell_1}, \mathcal{D}_{\tilde{\ell}}(i)$, $\mathcal{D}_{\tilde{\ell}}^{\mathrm{ur}*}$, or $\mathcal{D}_{\tilde{\ell}}^*$. Applying the element $g = (1, {}^t n(-x_{10}x_{20}^{-1}))$ to x, which is permissible by Proposition 5.3, we may assume that $x_{10} = 0$; note that this doesn't change $k_v(x)$. Further, applying

$$g = (a(x_{11}^{-1}, 1), \mathrm{N}_{\tilde{k}_v/k_v}(x_{11})a(1, x_{20}^{-1}))$$

we may also assume that $x_{11} = x_{20} = 1$. This implies that $a_0(x) = 1$ and

$$a_1(x) = \mathrm{Tr}_{\tilde{k}_v/k_v}(x_{21}) - x_{12}, \quad a_2(x) = \mathrm{N}_{\tilde{k}_v/k_v}(x_{21}) - x_{22}. \tag{27}$$

In case (5.4.4), $x_{22} \equiv 0\ (\mathfrak{p}_v^{\tilde{\delta}_v+1})$ and so $x_{22} \equiv 0\ (\mathfrak{p}_v^2)$. Note that $\tilde{\ell}+1, i+1, \tilde{\delta}_v \geqslant 2$ for (5.4.1)–(5.4.3), respectively, and so $x_{22} \equiv 0\ (\mathfrak{p}_v^2)$ in these cases also. Therefore,

$$\mathrm{ord}_{k_v}(a_2(x)) = \mathrm{ord}_{k_v}(\mathrm{N}_{\tilde{k}_v/k_v}(x_{21}) - x_{22}) = \mathrm{ord}_{k_v}(\mathrm{N}_{\tilde{k}_v/k_v}(x_{21})) = 1.$$

By this and the last conditions in (20)–(22), $F_x(z,1)$ is an Eisenstein polynomial such that the corresponding ℓ is $\ell_1, \tilde{\ell}, \tilde{\ell}$ for (5.4.1)–(5.4.3), respectively. In case (5.4.4), $x_{12} \equiv 0\ (\mathfrak{p}_v^{\tilde{\delta}_v+1})$ and so $\mathrm{ord}_{k_v}(a_1(x)) = \tilde{\ell}$ or $\mathrm{ord}_{k_v}(a_1(x)) \geqslant m_v + 1$, by Lemma 3.1. The first implication in (5.4.1) is now clear.

Consider (5.4.2). By assumption,

$$\mathrm{ord}_{k_v}(\mathrm{Tr}_{\tilde{k}_v/k_v}(x_{21}) - a_1(x)) = \mathrm{ord}_{k_v}(x_{12}) = i,$$

$$\mathrm{N}_{\tilde{k}_v/k_v}(x_{21}) - a_2(x) = x_{22} \equiv 0\ (\mathfrak{p}_v^{i+1}).$$

So, by Corollary 3.15, lev $(k_v(x), \tilde{k}_v) = i$. This proves the first implication of (5.4.2).

Consider (5.4.3). We have

$$\mathrm{Tr}_{\tilde{k}_v/k_v}(x_{21}) - a_1(x) = x_{12} \equiv 0\ (\mathfrak{p}_v^{\tilde{\delta}_v}),$$

$$\mathrm{N}_{\tilde{k}_v/k_v}(x_{21}) - a_2(x) = x_{22} \equiv 0\ (\mathfrak{p}_v^{\tilde{\delta}_v}).$$

Therefore $S_{\tilde{\delta}_v,\tilde{\delta}_v}(k_v(x), \tilde{k}_v) \neq \emptyset$. So the only possible types are (rm rm ur), (rm rm)*, by Corollary 3.9 and Proposition 3.10.

By similar considerations, $S_{\tilde{\delta}_v+1,\tilde{\delta}_v+1}(k_v(x), \tilde{k}_v) \neq \emptyset$ in case (5.4.4). So the only possible type is (rm rm)*, by the remark after Corollary 3.9.

We now consider the second implication of (5.4.1)–(5.4.3). Suppose the roots of an Eisenstein equation $p(z) = z^2 + a_1 z + a_2 = 0$ generate k'. In (5.4.1), there exists $\eta \in \tilde{k}_v$ such that $\mathrm{ord}_{k_v}(\mathrm{Tr}_{\tilde{k}_v}(\eta) - a_1) = \ell$ and $\mathrm{N}_{\tilde{k}_v}(\eta) - a_2 \equiv 0\ (\mathfrak{p}_v^{\ell+1})$, by Proposition 3.12. In (5.4.2), by Corollary 3.15, there exists $\eta \in \tilde{k}_v$ such that $\mathrm{ord}_{k_v}(\mathrm{Tr}_{\tilde{k}_v}(\eta) - a_1) = i$ and $\mathrm{N}_{\tilde{k}_v}(\eta) - a_2 \equiv 0\ (\mathfrak{p}_v^{i+1})$. In (5.4.3), by Proposition 3.10, there exists $\eta \in \tilde{k}_v$ such that $\mathrm{Tr}_{\tilde{k}_v}(\eta) - a_1, \mathrm{N}_{\tilde{k}_v}(\eta) - a_2 \in \mathfrak{p}_v^{\tilde{\delta}_v}$. Let $x = (n(\eta - a_1), 1)w_p$ in all cases. Then

$$x = \left(\begin{pmatrix} 0 & 1 \\ 1 & \mathrm{Tr}_{\tilde{k}_v}(\eta) - a_1 \end{pmatrix}, \begin{pmatrix} 1 & \eta^\sigma \\ \eta & \mathrm{N}_{\tilde{k}_v}(\eta) - a_2 \end{pmatrix} \right)$$

and so x satisfies (20), (21) or (22). This proves the second implication of (5.4.1)–(5.4.3). $\square$

Our next task is to prove that the sets defined in (20)–(24) are covered by the K_v-orbits of suitably chosen standard representatives.

Lemma 5.5. *Suppose p, p' are Eisenstein polynomials whose roots generate the same ramified quadratic field over k_v. Then there exists $\kappa \in K_v$ such that $w_p = \kappa w_{p'}$.*

Proof. Let α_1, α_2 and α_1', α_2' be roots of p, p', respectively. Since α_1, α_1' are both uniformizers of the same field, there exist $c \in \mathfrak{p}_v$, $d \in \mathcal{O}_v^\times$ such that $\alpha_1 = c + d\alpha_1'$. Let $\kappa_{\alpha,\alpha'} = n(-c)a(1,d)$. Then $\kappa_{\alpha,\alpha'} \in \mathrm{GL}(2)_{\mathcal{O}_v}$ and $h_\alpha = \kappa_{\alpha,\alpha'} h_{\alpha'}$. If $k_v(\alpha_1) \neq \tilde{k}_v$ then, by [3], (3.18),

$$\begin{aligned} w_p &= (h_\alpha, (\alpha_2 - \alpha_1)^{-1} h_\alpha) w \\ &= (\kappa_{\alpha,\alpha'}, d^{-1}\kappa_{\alpha,\alpha'})(h_{\alpha'}, (\alpha_2' - \alpha_1')^{-1} h_{\alpha'}) w \\ &= (\kappa_{\alpha,\alpha'}, d^{-1}\kappa_{\alpha,\alpha'}) w_{p'} \end{aligned}$$

and $(\kappa_{\alpha,\alpha'}, d^{-1}\kappa_{\alpha,\alpha'}) \in K_v$ since $d \in \mathcal{O}_v^\times$. If $k_v(\alpha_1) = \tilde{k}_v$ then

$$\begin{aligned} w_p &= (h_\alpha, h_\alpha, (\alpha_2 - \alpha_1)^{-1} h_\alpha) w \\ &= (\kappa_{\alpha,\alpha'}, \kappa_{\alpha,\alpha'}, d^{-1}\kappa_{\alpha,\alpha'}) w_{p'}. \end{aligned}$$

Note that we are regarding both the triples $(h_\alpha, h_\alpha, (\alpha_2 - \alpha_1)^{-1} h_\alpha)$ and $(\kappa_{\alpha,\alpha'}, \kappa_{\alpha,\alpha'}, d^{-1}\kappa_{\alpha,\alpha'})$ as elements of $G_{\tilde{k}_v}$ here. Since $c, d \in k_v$, the triple $(\kappa_{\alpha,\alpha'}, \kappa_{\alpha,\alpha'}, d^{-1}\kappa_{\alpha,\alpha'})$ is an element of the group G_{k_v} regarded as embedded in the group $G_{\tilde{k}_v}$. Therefore, the triple $(\kappa_{\alpha,\alpha'}, \kappa_{\alpha,\alpha'}, d^{-1}\kappa_{\alpha,\alpha'})$ is an element of K_v in this case also. □

If $\ell_1 \neq \tilde{\ell}$, we choose Eisenstein polynomials $p_{\ell_1,j}$, for $j = 1, \ldots, N_{\ell_1}$, so that $\{k_v(w_{p_{\ell_1,j}})\}$ is a complete set of representatives for the classes in $\mathfrak{X}_{\ell_1}$. Similarly, we choose Eisenstein polynomials $p_{\tilde{\ell},i,j}$, for $j = 1, \ldots, N_{\tilde{\ell}}(i)$, so that $\{k_v(w_{p_{\tilde{\ell},i,j}})\}$ is a complete set of representatives for the classes in $\mathfrak{X}_{\tilde{\ell}}(i)$ and an Eisenstein polynomial $p_{\tilde{\ell}}^{\mathrm{ur}}$ so that $\mathfrak{X}_{\tilde{\ell}}^{\mathrm{ur}}$ is the singleton containing the class of $k_v(w_{p_{\tilde{\ell}}^{\mathrm{ur}}})$. In order to simplify the notation, we write $w_{\ell_1,j}$ in place of $w_{p_{\ell_1,j}}$, $w_{\tilde{\ell},i,j}$ in place of $w_{p_{\tilde{\ell},i,j}}$ and $w_{\tilde{\ell}}^{\mathrm{ur}}$ in place of $w_{p_{\tilde{\ell}}^{\mathrm{ur}}}$.

Proposition 5.6.

(5.6.1) *If $x \in \mathscr{D}_{\ell_1}$ then $x \in \cup_j K_v w_{\ell_1,j}$.*
(5.6.2) *If $x \in \mathscr{D}_{\tilde{\ell}}(i)$ then $x \in \cup_j K_v w_{\tilde{\ell},i,j}$.*
(5.6.3) *If $x \in \mathscr{D}_{\tilde{\ell}}^{\mathrm{ur}*}$ then $x \in K_v w_{\tilde{\ell}}^{\mathrm{ur}} \cup K_v w_\eta$.*
(5.6.4) *If $x \in \mathscr{D}_{\tilde{\ell}}^{*}$ then $x \in K_v w_\eta$.*

Proof. As shown in the proof of Proposition 5.4, we may assume that $x_{10} = 0$ and $x_{11} = x_{20} = 1$. Let $p(z) = z^2 + a_1(x)z + a_2(x)$. Then, by (27),

$$x = (n(x_{21}^\sigma - a_1(x)), 1) w_p \in K_v w_p.$$

Consider (5.6.1). By Proposition 5.4 there exists j such that $k_v(x) = k_v(w_{\ell_1,j})$. By Lemma 5.5, $w_p \in K_v w_{\ell_1,j}$ and so $x \in K_v w_{\ell_1,j}$. Cases (5.6.2), (5.6.3) and (5.6.4) are similar. □

Next we shall find the volume of the sets defined in (20)–(24) and find the number of copies needed to cover the K_v-orbits of the standard representatives.

Lemma 5.7. *Suppose* $u_1, u_2 \in \tilde{\mathcal{O}}_v$, $\mathrm{ord}_{\tilde{k}_v}(u_1) = j$, *and* $\mathrm{ord}_{\tilde{k}_v}(u_2) \geqslant j+1$.

(5.7.1) *If* $j < \tilde{\delta}_v$ *then* $\mathrm{ord}_{k_v}(\mathrm{N}_{\tilde{k}_v/k_v}(u_1)) < \mathrm{ord}_{k_v}(\mathrm{Tr}_{\tilde{k}_v/k_v}(u_2))$.

(5.7.2) *If* $j = \tilde{\delta}_v$ *or* $\tilde{\delta}_v + 1$ *then* $\mathrm{N}_{\tilde{k}_v/k_v}(u_1), \mathrm{Tr}_{\tilde{k}_v/k_v}(u_2) \in \mathfrak{p}_v^j$.

Proof. By Lemma 3.2, $\mathrm{ord}_{k_v}(\mathrm{Tr}_{\tilde{k}_v/k_v}(u_2)) \geqslant \lfloor (j+\tilde{\delta}_v+1)/2 \rfloor \geqslant (j+\tilde{\delta}_v)/2$. If $j < \tilde{\delta}_v$ then $j < (j+\tilde{\delta}_v)/2$ and, since $\mathrm{ord}_{k_v}(\mathrm{N}_{\tilde{k}_v/k_v}(u_1)) = j$, (5.7.1) follows. In (5.7.2), it is clear that $\mathrm{N}_{\tilde{k}_v/k_v}(u_1) \in \mathfrak{p}_v^j$. If $j = \tilde{\delta}_v$ then Lemma 3.2 gives $\mathrm{ord}_{k_v}(\mathrm{Tr}_{\tilde{k}_v/k_v}(u_2)) \geqslant \tilde{\delta}_v$, and if $j = \tilde{\delta}_v + 1$ then it gives $\mathrm{ord}_{k_v}(\mathrm{Tr}_{\tilde{k}_v/k_v}(u_2)) \geqslant \tilde{\delta}_v + 1$. This completes the verification of (5.7.2). □

Proposition 5.8.

(5.8.1) *Suppose* $\ell_1 \neq \tilde{\ell}$ *and* $\ell_1 \leqslant m_v$ *and let* $\ell = \min\{\ell_1, \tilde{\ell}\}$. *Then*

$$\mathrm{vol}(\mathscr{D}_{\ell_1}) = q_v^{-\ell_1-\ell-2}(1-q_v^{-1})^4.$$

(5.8.2) *Suppose* $\ell = \tilde{\ell} \leqslant m_v$ *and* $\ell_1 = m_v + 1$. *Then*

$$\mathrm{vol}(\mathscr{D}_{\ell_1}) = q_v^{-\ell_1-\ell-2}(1-q_v^{-1})^3.$$

(5.8.3) *Suppose* $\tilde{\ell} \leqslant m_v$. *Then*

$$\mathrm{vol}(\mathscr{D}_{\tilde{\ell}}(\tilde{\ell})) = q_v^{-2\tilde{\ell}-2}(1-q_v^{-1})^3(1-2q_v^{-1}).$$

(5.8.4) *Suppose* $\tilde{\ell} < i < \tilde{\delta}_v$ *or* $\tilde{\ell} = m_v + 1$. *Then*

$$\mathrm{vol}(\mathscr{D}_{\tilde{\ell}}(i)) = q_v^{-2i-2}(1-q_v^{-1})^4.$$

(5.8.5) *We have* $\mathrm{vol}(\mathscr{D}_{\tilde{\ell}}^{\mathrm{ur}\,*}) = q_v^{-2\tilde{\delta}_v-1}(1-q_v^{-1})^3$.

(5.8.6) *We have* $\mathrm{vol}(\mathscr{D}_{\tilde{\ell}}^{*}) = q_v^{-8(\tilde{\delta}_v+1)}$.

Proof. Part (5.8.6) is obvious.

Consider (5.8.1) and (5.8.2). Suppose $\ell_1 < \tilde{\ell}$ and $x \in V_{\mathcal{O}_v}$ satisfies condition (20) except possibly for the last condition. Then $a_1(x) \equiv \mathrm{Tr}_{\tilde{k}_v/k_v}(x_{11}x_{21}^{\sigma}) - x_{12}x_{20}\ (\mathfrak{p}_v^{\ell+1})$ by (5) of Section 2. Since $\mathfrak{p}_v^{\tilde{\ell}} \subseteq \mathfrak{p}_v^{\ell+1}$, $\mathrm{Tr}_{\tilde{k}_v/k_v}(x_{11}x_{21}^{\sigma}) \equiv 0\ (\mathfrak{p}^{\ell+1})$, by Lemma 3.1, and $\mathrm{ord}_{k_v}(x_{12}x_{20}) = \ell$. Thus the last condition of (20) is automatically satisfied. The volumes of the sets of $x_{10}, x_{11}, x_{12}, x_{20}, x_{21}, x_{22}$ satisfying condition (20) are $1, 1 - q_v^{-1}, q_v^{-\ell}(1-q_v^{-1}), 1-q_v^{-1}, q_v^{-1}(1-q_v^{-1}), q_v^{-\ell-1}$, respectively. Therefore,

$$\mathrm{vol}(\mathscr{D}_{\ell_1}) = q_v^{-2\ell-2}(1-q_v^{-1})^4 = q_v^{-\ell_1-\ell-2}(1-q_v^{-1})^4.$$

Now suppose that $\ell_1 > \tilde{\ell}$ and again assume that $x \in V_{\mathcal{O}_v}$ satisfies the conditions of (20) except possibly for the last. We have $\tilde{\ell} \leqslant m_v$ and so, by Lemma 3.1, $\mathrm{ord}_{k_v}(\mathrm{Tr}_{\tilde{k}_v/k_v}(x_{11}x_{21}^{\sigma})) = \tilde{\ell}$. Since $\ell = \tilde{\ell}$, $\mathrm{ord}_{k_v}(x_{10}x_{22}) \geqslant \tilde{\ell} + 1$ and so

$$\mathrm{ord}_{k_v}(\mathrm{Tr}_{\tilde{k}_v/k_v}(x_{11}x_{21}^{\sigma}) - x_{10}x_{22}) = \tilde{\ell}.$$

If $\ell_1 \leqslant m_v$ then it follows from this and (5) of Section 2 that x satisfies the last condition of (20) if and only if

$$x_{12} \equiv -x_{20}^{-1}(\mathrm{Tr}_{\tilde{k}_v/k_v}(x_{11}x_{21}^{\sigma}) - x_{10}x_{22})\ (\mathfrak{p}_v^{\ell_1}) \tag{28}$$

but the corresponding congruence with $\ell_1 + 1$ in place of ℓ_1 is false. With the other variables fixed, the volume of the set of x_{12} satisfying (28) is $q_v^{-\ell_1}$ and hence the volume of the set of allowable x_{12} is $q_v^{-\ell_1} - q_v^{-(\ell_1+1)} = q_v^{-\ell_1}(1-q_v^{-1})$. This gives

$$\begin{aligned}\mathrm{vol}(\mathscr{D}_{\ell_1}) &= (1-q_v^{-1})(q_v^{-\ell_1}(1-q_v^{-1}))(1-q_v^{-1})(q_v^{-1}(1-q_v^{-1}))q_v^{-(\ell+1)} \\ &= q_v^{-\ell_1-\ell-2}(1-q_v^{-1})^4\end{aligned}$$

in this case. If $\ell_1 = m_v + 1$ the reasoning is the same, except that (28) is the only condition on x_{12}. We thus obtain a similar formula for $\mathrm{vol}(\mathscr{D}_{\ell_1})$ with one fewer factors of $(1-q_v^{-1})$. This proves (5.8.1) and (5.8.2).

Consider (5.8.3). Suppose $\tilde{\ell} \leqslant m_v$ and x satisfies the conditions of (21) for $i = \tilde{\ell}$ except possibly for the last condition. Since $x_{10}x_{22} \in \mathfrak{p}_v^{\tilde{\ell}+1}$,

$$a_1(x) \equiv \mathrm{Tr}_{\tilde{k}_v/k_v}(x_{11}x_{21}^{\sigma}) - x_{12}x_{20}\ (\mathfrak{p}_v^{\tilde{\ell}+1}) \tag{29}$$

by (5) of Section 2. The order of the first term is $\tilde{\ell}$, by Lemma 3.1, and the order of x_{12} is $\tilde{\ell}$. So, when x_{11}, x_{21}, x_{12} are fixed, for $a_1(x)$ to be of order $\tilde{\ell}$, x_{20} has to be a unit which is not congruent to $\mathrm{Tr}_{\tilde{k}_v/k_v}(x_{11}x_{21}^{\sigma})x_{12}^{-1}$ modulo $\mathfrak{p}_v$. Therefore,

$$\begin{aligned}\mathrm{vol}(\mathscr{D}_{\tilde{\ell}}(\tilde{\ell})) &= (1-q_v^{-1})(q_v^{-\tilde{\ell}}(1-q_v^{-1}))(1-2q_v^{-1})(q_v^{-1}(1-q_v^{-1}))q_v^{-\tilde{\ell}-1}\\ &= q_v^{-2\tilde{\ell}-2}(1-q_v^{-1})^3(1-2q_v^{-1}).\end{aligned}$$

Consider (5.8.4). Suppose that $x\in V_{\mathcal{O}_v}$ satisfies all the conditions of (21) except possibly for the last. We shall show that the last condition follows automatically. First suppose that $\tilde{\ell}\leqslant m_v$ and $\tilde{\ell}<i<\tilde{\delta}_v$. Then $\mathrm{ord}_{k_v}(\mathrm{Tr}_{\tilde{k}_v/k_v}(x_{11}x_{21}^{\sigma}))=\tilde{\ell}$, by Lemma 3.1, and $x_{10}x_{22}, x_{12}x_{20}\in\mathfrak{p}_v^i\subseteq\mathfrak{p}_v^{\tilde{\ell}+1}$. Thus, by (5), $\mathrm{ord}_{k_v}(a_1(x))=\tilde{\ell}$, as claimed. Now suppose that $\tilde{\ell}=m_v+1$ and $\tilde{\ell}\leqslant i<\tilde{\delta}_v$. Then $\mathrm{ord}_{k_v}(\mathrm{Tr}_{\tilde{k}_v/k_v}(x_{11}x_{21}^{\sigma}))\geqslant\tilde{\ell}$, by Lemma 3.1, and $x_{10}x_{22}, x_{12}x_{20}\in\mathfrak{p}_v^i\subseteq\mathfrak{p}_v^{\tilde{\ell}}$. Thus, by (5) of Section 2, $\mathrm{ord}_{k_v}(a_1(x))\geqslant\tilde{\ell}$ and again the last condition holds. This implies that

$$\begin{aligned}\mathrm{vol}(\mathscr{D}_{\tilde{\ell}}(i)) &= (1-q_v^{-1})(q_v^{-i}(1-q_v^{-1}))(1-q_v^{-1})(q_v^{-1}(1-q_v^{-1}))q_v^{-i-1}\\ &= q_v^{-2i-2}(1-q_v^{-1})^4.\end{aligned}$$

Consider (5.8.5). Suppose x satisfies condition (22) except possibly for the last condition. Then $x_{12}, x_{22}\in\mathfrak{p}_v^{\tilde{\delta}_v}\subseteq\mathfrak{p}_v^{\tilde{\ell}+1}$. Since we know that $\mathrm{ord}_{k_v}(\mathrm{Tr}_{\tilde{k}_v/k_v}(x_{11}x_{21}^{\sigma}))$ is $=\tilde{\ell}$ if $\tilde{\ell}\leqslant m_v$ and is $\geqslant m_v+1$ if $\tilde{\ell}=m_v+1$, the last condition of (22) is always satisfied. Therefore,

$$\begin{aligned}\mathrm{vol}(\mathscr{D}_{\tilde{\ell}}^{\mathrm{ur}\,*}) &= (1-q_v^{-1})q_v^{-\tilde{\delta}_v}(1-q_v^{-1})(q_v^{-1}(1-q_v^{-1}))q_v^{-\tilde{\delta}_v}\\ &= q_v^{-2\tilde{\delta}_v-1}(1-q_v^{-1})^3.\end{aligned}$$

This finishes all the cases. □

Proposition 5.9. (5.9.1) *Let $\ell_1\neq\tilde{\ell}$ and $\ell=\min\{\ell_1,\tilde{\ell}\}$. Suppose $g\in K_v$, $x,y\in\mathscr{D}_{\ell_1}$ and $gx=y$. Then $g\in H(\ell)$.*

(5.9.2) *Let $\tilde{\ell}\leqslant i<\tilde{\delta}_v$. Suppose $g\in K_v, x,y\in\mathscr{D}_{\tilde{\ell}}(i)$ and $gx=y$. Then $g\in H(i)$.*

(5.9.3) *Suppose $g\in K_v$, $x,y\in\mathscr{D}_{\tilde{\ell}}^{\mathrm{ur}\,*}$ and $gx=y$. Then $g\in H(\tilde{\delta}_v-1)$.*

Proof. Suppose $g=(g_1,g_2)$ is as in (3) of Section 2. Since both $F_x(v)$ and $F_y(v)$ are congruent to unit scalar multiples of v_1^2 modulo $\mathfrak{p}_v$, $g_{221}\equiv 0\ (\mathfrak{p}_v)$. Since $(1,g_2)\in H(j)$ for every $j\geqslant 0$, we may assume that $g_2=1$.

Since $x_{20}, y_{20}\in\mathcal{O}_v^{\times}$, $x_{21}, y_{21}\in\tilde{\mathfrak{p}}_v$ and $x_{22}, y_{22}\in\mathfrak{p}_v$, we have $g_{121}\equiv 0\ (\tilde{\mathfrak{p}}_v)$. This implies that $g_{111}, g_{122}\in\tilde{\mathcal{O}}_v^{\times}$. By (26),

$$y_{22}=\mathrm{N}_{\tilde{k}_v/k_v}(g_{121})x_{20}+\mathrm{Tr}_{\tilde{k}_v/k_v}(g_{121}g_{122}^{\sigma}x_{21})+\mathrm{N}_{\tilde{k}_v/k_v}(g_{122})x_{22}.$$

Consider (5.9.1). Since $\ell < \tilde{\delta}_v$, if $\mathrm{ord}_{\tilde{k}_v}(g_{121}) \leqslant \ell$ then

$$\mathrm{ord}_{k_v}(\mathrm{Tr}_{\tilde{k}_v/k_v}(g_{121}g_{122}^{\sigma}x_{21})) > \mathrm{ord}_{k_v}(\mathrm{N}_{\tilde{k}_v/k_v}(g_{121})x_{20}) = \ell$$

by Lemma 5.7. Since $x_{22}, y_{22} \in \mathfrak{p}_v^{\ell+1}$, this is a contradiction. Since $i, \tilde{\delta}_v - 1 < \tilde{\delta}_v$, (5.9.2) and (5.9.3) are similar. □

The next corollary follows easily from Lemma 5.3 and Proposition 5.9.

Corollary 5.10. (5.10.1) *Suppose* $\ell_1 \neq \tilde{\ell}$. *If* $g, g' \in K_v$ *and* $g\mathscr{D}_{\ell_1} \cap g'\mathscr{D}_{\ell_1} \neq \emptyset$ *then* $g\mathscr{D}_{\ell_1} = g'\mathscr{D}_{\ell_1}$.

(5.10.2) *Suppose* $\tilde{\ell} \leqslant i < \tilde{\delta}_v$. *If* $g, g' \in K_v$ *and* $g\mathscr{D}_{\tilde{\ell}}(i) \cap g'\mathscr{D}_{\tilde{\ell}}(i) \neq \emptyset$ *then* $g\mathscr{D}_{\tilde{\ell}}(i) = g'\mathscr{D}_{\tilde{\ell}}(i)$.

(5.10.3) *If* $g, g' \in K_v$ *and* $g\mathscr{D}_{\tilde{\ell}}^{\mathrm{ur}*} \cap g'\mathscr{D}_{\tilde{\ell}}^{\mathrm{ur}*} \neq \emptyset$ *then* $g\mathscr{D}_{\tilde{\ell}}^{\mathrm{ur}*} = g'\mathscr{D}_{\tilde{\ell}}^{\mathrm{ur}*}$.

We are now ready to calculate $\sum_x \mathrm{vol}(K_v x)$.

Proposition 5.11.

(5.11.1) *Suppose* $\ell_1 \neq \tilde{\ell}$ *and* $\ell_1 \leqslant m_v$. *Then*

$$\sum_j \mathrm{vol}(K_v w_{\ell_1,j}) = q_v^{-\ell_1}(1-q_v^{-1})^2(1-q_v^{-2})^2.$$

(5.11.2) *Suppose* $\tilde{\ell} \leqslant m_v$ *and* $\ell_1 = m_v + 1$. *Then*

$$\sum_j \mathrm{vol}(K_v w_{\ell_1,j}) = q_v^{-\ell_1}(1-q_v^{-1})(1-q_v^{-2})^2.$$

(5.11.3) *Suppose* $\tilde{\ell} \leqslant m_v$. *Then*

$$\sum_j \mathrm{vol}(K_v w_{\tilde{\ell},\tilde{\ell},j}) = q_v^{-\tilde{\ell}}(1-q_v^{-1})(1-2q_v^{-1})(1-q_v^{-2})^2.$$

(5.11.4) *Suppose* $\tilde{\ell} < i < \tilde{\delta}_v$ *or* $\tilde{\ell} = m_v + 1$. *Then*

$$\sum_j \mathrm{vol}(K_v w_{\tilde{\ell},i,j}) = q_v^{-i}(1-q_v^{-1})^2(1-q_v^{-2})^2.$$

(5.11.5) $\mathrm{vol}(K_v w_{\tilde{\ell}}^{\mathrm{ur}}) + \mathrm{vol}(K_v w_\eta) = q_v^{-\tilde{\delta}_v}(1-q_v^{-1})(1-q_v^{-2})^2.$

Proof. Consider (5.1.1). Let $\ell = \min\{\ell_1, \tilde{\ell}\}$. By Propositions 5.6 and 5.9 and Corollary 5.10, $\bigcup_j K_v w_{\ell_1,j}$ is a disjoint union of translates of $\mathscr{D}_{\ell_1}$ and the number of translates is $Q(\ell)$. So, by (25) and Proposition 5.8.1,

$$\sum_j \mathrm{vol}(K_v w_{\ell_1,j}) = Q(\ell)\,\mathrm{vol}(\mathscr{D}_{\ell_1}) = q_v^{\ell+2}(1+q_v^{-1})^2 q_v^{-\ell_1-\ell-2}(1-q_v^{-1})^4$$
$$= q_v^{-\ell_1}(1-q_v^{-1})^2(1-q_v^{-2})^2.$$

Cases (5.11.2)–(5.11.5) are similar using Proposition 5.8.2–5.8.5. $\square$

Our next task is to determine $\mathrm{vol}(K_v w_\eta)$. Let $\tilde{p}(z)$ be the polynomial introduced in the second paragraph of this section and recall, as noted after (23), that $w_\eta = (n(\eta_2),1)w_{\tilde{p}}$. Let $G_{w_\eta\ \mathcal{O}_v/\mathfrak{p}_v^{\tilde{\delta}_v+1}}$ and $G^\circ_{w_\eta\ \mathcal{O}_v/\mathfrak{p}_v^{\tilde{\delta}_v+1}}$ be the sets of $(\mathcal{O}_v/\mathfrak{p}_v^{\tilde{\delta}_v+1})$-valued points of the schemes G_{w_η} and $G^\circ_{w_\eta}$ over $\mathcal{O}_v$.

Lemma 5.12. *The order of* $G^\circ_{w_\eta \mathcal{O}_v/\mathfrak{p}_v^{\tilde{\delta}_v+1}}$ is $q_v^{4\tilde{\delta}_v+3}(q_v-1)$.

Proof. Since $w_\eta = (n(\eta_2),1)w_{\tilde{p}}$ and $(n(\eta_2),1)\in K_v$, $G^\circ_{w_\eta\ \mathcal{O}_v/\mathfrak{p}_v^{\tilde{\delta}_v+1}}$ and the similarly defined set $G^\circ_{w_{\tilde{p}}\ \mathcal{O}_v/\mathfrak{p}_v^{\tilde{\delta}_v+1}}$ are conjugate within $G_{\mathcal{O}_v/\mathfrak{p}_v^{\tilde{\delta}_v+1}}$ and so it suffices to find the order of $G_{w^\circ_{\tilde{p}}\ \mathcal{O}_v/\mathfrak{p}_v^{\tilde{\delta}_v+1}}$. Let

$$A_{\tilde{p}}(c,d) = \begin{pmatrix} c & -d \\ b_1 d & c - b_1 d \end{pmatrix}.$$

It was proved in [2], Lemma 4.2 that $G^\circ_{w_{\tilde{p}}\mathcal{O}_v/\mathfrak{p}_v^{\tilde{\delta}_v+1}}$ consists of elements of the form

$$(A_{\tilde{p}}(c_1,d_1), A_{\tilde{p}}(c_2,d_2))$$

where $c_1, d_1 \in \tilde{\mathcal{O}}_v/\tilde{\mathfrak{p}}_v^{2(\tilde{\delta}_v+1)}$, $c_2, d_2 \in \mathcal{O}_v/\mathfrak{p}_v^{\tilde{\delta}_v+1}$, $\det A_{\tilde{p}}(c_1,d_1) \in (\tilde{\mathcal{O}}_v/\tilde{\mathfrak{p}}_v^{2(\tilde{\delta}_v+1)})^\times$ and c_2 and d_2 are related to c_1 and d_1 by the equation

$$A_{\tilde{p}}(c_2,d_2) = A_{\tilde{p}}(c_1,d_1)^{-1} A_{\tilde{p}}(c_1^\sigma, d_1^\sigma)^{-1}.$$

Note that $\det A_{\tilde{p}}(c_1,d_1) \in (\tilde{\mathcal{O}}_v/\tilde{\mathfrak{p}}_v^{2(\tilde{\delta}_v+1)})^\times$ if and only if $c_1 \in (\tilde{\mathcal{O}}_v/\tilde{\mathfrak{p}}_v^{2(\tilde{\delta}_v+1)})^\times$. The expression for the order follows immediately. $\square$

We denote by $\bar{w}_\eta$ the reduction of w_η modulo $\mathfrak{p}_v^{\tilde{\delta}_v+1}$ and by $G_{\bar{w}_\eta\ \mathcal{O}_v/\mathfrak{p}_v^{\tilde{\delta}+1}}$ the stabilizer of $\bar{w}_\eta$ in $G_{\mathcal{O}_v/\mathfrak{p}_v^{\tilde{\delta}_v+1}}$. Clearly $G_{w_\eta\ \mathcal{O}_v/\mathfrak{p}_v^{\tilde{\delta}_v+1}}$ is a subgroup of $G_{\bar{w}_\eta\ \mathcal{O}_v/\mathfrak{p}_v^{\tilde{\delta}_v+1}}$.

Lemma 5.13. *We have* $[G_{\bar{w}_\eta\ \mathcal{O}_v/\mathfrak{p}_v^{\tilde{\delta}_v+1}} : G^\circ_{w_\eta\ \mathcal{O}_v/\mathfrak{p}_v^{\tilde{\delta}_v+1}}] = 2q_v^{\tilde{\delta}_v+2\lfloor\tilde{\delta}_v/2\rfloor}$.

Proof. Let $\bar{w}_{\tilde{p}}$ denote the reduction of $w_{\tilde{p}}$ modulo $\mathfrak{p}_v^{\tilde{\delta}_v+1}$ and $G_{\bar{w}_{\tilde{p}}\,\mathcal{O}_v/\mathfrak{p}_v^{\tilde{\delta}_v+1}}$ the stabilizer of $\bar{w}_{\tilde{p}}$ in $G_{\mathcal{O}_v/\mathfrak{p}_v^{\tilde{\delta}_v+1}}$. Our first step will be to show that every right $G^{\circ}_{w_{\tilde{p}}\,\mathcal{O}_v/\mathfrak{p}_v^{\tilde{\delta}_v+1}}$ coset in $G_{\bar{w}_{\tilde{p}}\,\mathcal{O}_v/\mathfrak{p}_v^{\tilde{\delta}_v+1}}$ has a representative of the particular form given in (30) below.

For $x=(x_1,x_2)\in V_{\mathcal{O}_v/\mathfrak{p}_v^{\tilde{\delta}_v+1}}$, let $\mathrm{Span}(x)$ be the $(\mathcal{O}_v/\mathfrak{p}_v^{\tilde{\delta}_v+1})$-module spanned by x_1 and x_2 inside $W_{\mathcal{O}_v/\mathfrak{p}_v^{\tilde{\delta}_v+1}}$. As was stated in [2], Lemma 4.4, if $g_1\in \mathrm{GL}(2)_{\tilde{\mathcal{O}}_v/\tilde{\mathfrak{p}}_v^{2(\tilde{\delta}_v+1)}}$ then there exists $g_2\in\mathrm{GL}(2)_{\mathcal{O}_v/\mathfrak{p}_v^{\tilde{\delta}_v+1}}$ such that $(g_1,g_2)\in G_{x\mathcal{O}_v/\mathfrak{p}_v^{\tilde{\delta}_v+1}}$ if and only if $\mathrm{Span}((g_1,1)x)=\mathrm{Span}(x)$.

Suppose that $g=(g_1,g_2)\in G_{\bar{w}_{\tilde{p}}\,\mathcal{O}_v/\mathfrak{p}_v^{\tilde{\delta}_v+1}}$. Since $F_{w_{\tilde{p}}}(v_1,v_2)$ reduces to v_1^2 modulo $\mathfrak{p}_v$, $g_{221}\in(\mathfrak{p}_v/\mathfrak{p}_v^{\tilde{\delta}_v+1})$. Using this fact and examining the second component of $w_{\tilde{p}}$ modulo $\mathfrak{p}_v$, we find that $g_{121}\in(\tilde{\mathfrak{p}}_v/\tilde{\mathfrak{p}}_v^{2(\tilde{\delta}_v+1)})$. This implies that g_{111} and g_{122} lie in $(\tilde{\mathcal{O}}_v/\tilde{\mathfrak{p}}_v^{2(\tilde{\delta}_v+1)})^{\times}$. If we put $c_1=g_{122}$, $d_1=g_{112}$ and $A_{\tilde{p}}(c_2,d_2)=A_{\tilde{p}}(c_1,d_1)^{-1}A_{\tilde{p}}(c_1^{\sigma},d_1^{\sigma})^{-1}$ then $(A_{\tilde{p}}(c_1,d_1),A_{\tilde{p}}(c_2,d_2))\in G^{\circ}_{w_{\tilde{p}}\,\mathcal{O}_v/\mathfrak{p}_v^{\tilde{\delta}_v+1}}$, the $(1,2)$-entry of $A_{\tilde{p}}(c_1,d_1)g_1$ is 0 and the (1,1)-entry is $\det(g_1)$. Since $\det(g_1)\in(\tilde{\mathcal{O}}_v/\tilde{\mathfrak{p}}_v^{2(\tilde{\delta}_v+1)})^{\times}$, we may further multiply on the left by $(A_{\tilde{p}}(\det(g_1)^{-1},0),*)\in G^{\circ}_{w_{\tilde{p}}\,\mathcal{O}_v/\mathfrak{p}_v^{\tilde{\delta}_v+1}}$ to find a representative for the right $G^{\circ}_{w_{\tilde{p}}\,\mathcal{O}_v/\mathfrak{p}_v^{\tilde{\delta}_v+1}}$ coset of g having the form

$$\left(\begin{pmatrix}1 & 0\\ u & t\end{pmatrix},*\right) \tag{30}$$

with $t\in(\tilde{\mathcal{O}}_v/\tilde{\mathfrak{p}}_v^{2(\tilde{\delta}_v+1)})^{\times}$ and $u\in\tilde{\mathcal{O}}_v/\tilde{\mathfrak{p}}_v^{2(\tilde{\delta}_v+1)}$. It is easy to check that each coset has exactly one representative in this form.

Since $w_{\eta}=(n(\eta_2),1)w_{\tilde{p}}$ and $(n(\eta_2),1)\in K_v$, it easily follows that every right $G^{\circ}_{w_{\eta}\,\mathcal{O}_v/\mathfrak{p}_v^{\tilde{\delta}_v+1}}$ coset in $G_{\bar{w}_{\eta}\,\mathcal{O}_v/\mathfrak{p}_v^{\tilde{\delta}_v+1}}$ also has a unique representative in the form (30). Our second step will be to determine when such an element actually lies in $G_{\bar{w}_{\eta}\,\mathcal{O}_v/\mathfrak{p}_v^{\tilde{\delta}_v+1}}$. Suppose that (g_1,g_2) is in the form (30). Then $(g_1,g_2)\in G_{\bar{w}_{\eta}\,\mathcal{O}_v/\mathfrak{p}_v^{\tilde{\delta}_v+1}}$ if and only if $\mathrm{Span}((g_1,1)\bar{w}_{\eta})=\mathrm{Span}(\bar{w}_{\eta})$. Computation gives $(g_1,1)\bar{w}_{\eta}=(M_1,M_2)$ where

$$M_1=\begin{pmatrix}0 & t^{\sigma}\\ t & tu^{\sigma}+t^{\sigma}u\end{pmatrix}\quad M_2=\begin{pmatrix}1 & u^{\sigma}-t^{\sigma}\eta_2\\ u-t\eta_1 & uu^{\sigma}-t\eta_1u^{\sigma}-t^{\sigma}\eta_2u\end{pmatrix}.$$

Note that $y=\begin{pmatrix}y_0 & y_1\\ y_1^{\sigma} & y_2\end{pmatrix}$ is in $\mathrm{Span}(\bar{w}_{\eta})$ if and only if $y_2=0$ and $y_1-y_1^{\sigma}=y_0(\eta_1-\eta_2)$. Thus our element lies in the stabilizer of $\bar{w}_{\eta}$ if and only if

$$\begin{aligned}&tu^{\sigma}+t^{\sigma}u=t^{\sigma}-t=0,\\ &uu^{\sigma}-t\eta_1u^{\sigma}-t^{\sigma}\eta_2u=0,\\ &(u^{\sigma}-t^{\sigma}\eta_2)-(u-t\eta_1)=\eta_1-\eta_2.\end{aligned} \tag{31}$$

Since t must be a unit, the first equation is equivalent to $t = t^\sigma$, $u^\sigma = -u$. Using this, the second two equations become $u^2 = t(\eta_1 - \eta_2)u$ and $2u = (\eta_1 - \eta_2)(t-1)$. Making use of the second of these, the first is equivalent to $u(u + (\eta_1 - \eta_2)) = 0$. Thus (31) is equivalent to the system

$$t = t^\sigma, \quad u^\sigma = -u, \quad u(u + (\eta_1 - \eta_2)) = 0, \quad 2u = (\eta_1 - \eta_2)(t-1). \tag{32}$$

In the analysis of this system it will be convenient to adopt the usual abuse of notation by which classes in $\tilde{\mathcal{O}}_v/\tilde{\mathfrak{p}}_v^{2(\tilde{\delta}_v+1)}$ and their representatives in $\tilde{\mathcal{O}}_v$ are denoted by the same symbol.

Since $\mathrm{ord}_{\tilde{k}_v}(\eta_1 - \eta_2) = \tilde{\delta}_v$, the third equation in (32) is equivalent to the condition that either $\mathrm{ord}_{\tilde{k}_v}(u) \geqslant \tilde{\delta}_v + 2$ or $\mathrm{ord}_{\tilde{k}_v}(u + (\eta_1 - \eta_2)) \geqslant \tilde{\delta}_v + 2$. These two possibilities are mutually exclusive and it is easy to check that the bijection $(u,t) \mapsto (u - (\eta_1 - \eta_2), t-2)$ carries the set of solutions to (32) satisfying the first inequality onto the set of solutions satisfying the second. Thus we may assume henceforth that $\mathrm{ord}_{\tilde{k}_v}(u) \geqslant \tilde{\delta}_v + 2$ provided we then double the number of solutions found. Since $\mathrm{ord}_{\tilde{k}_v}(u) \geqslant \tilde{\delta}_v + 2$, $\mathrm{ord}_{k_v}(u + u^\sigma) \geqslant \lfloor (2\tilde{\delta}_v + 2)/2 \rfloor = \tilde{\delta}_v + 1$, by Lemma 3.2, and so $\mathrm{ord}_{\tilde{k}_v}(u + u^\sigma) \geqslant 2\tilde{\delta}_v + 2$. Thus the second equation in (32) is a consequence of the third and may be deleted from the system.

Now suppose that $\tilde{\delta}_v \leqslant 2m_v$, so that $\tilde{\delta}_v = 2\tilde{\ell}$. Since $\mathrm{ord}_{\tilde{k}_v}(u) \geqslant \tilde{\delta}_v + 2$, we may write $u = (\eta_1 - \eta_2)\pi_v \bar{u}$ with $\bar{u} \in \tilde{\mathcal{O}}_v$. The fourth equation in (32) is then equivalent to $t \equiv 1 + 2\pi_v \bar{u}\ (\tilde{\mathfrak{p}}_v^{\tilde{\delta}_v+2})$. Thus $t = 1 + 2\pi_v\bar{u} + \pi_v^{\tilde{\ell}+1}c$ with $c \in \tilde{\mathcal{O}}_v$. It follows that $t - t^\sigma = 2\pi_v(\bar{u} - \bar{u}^\sigma) + \pi_v^{\tilde{\ell}}(c - c^\sigma)$. But $(\bar{u} - \bar{u}^\sigma), (c - c^\sigma) \in \tilde{\mathfrak{p}}_v^{\tilde{\delta}_v}$ and so

$$t - t^\sigma \in \tilde{\mathfrak{p}}_v^{2m_v+2+\tilde{\delta}_v} + \tilde{\mathfrak{p}}_v^{2\tilde{\ell}+2+\tilde{\delta}_v} \subseteq \mathfrak{p}_v^{2\tilde{\delta}_v+2}.$$

Thus the first equation of (32) is a consequence of the third and fourth. There are thus $\#\ (\tilde{\mathfrak{p}}_v^{\tilde{\delta}_v+2}/\tilde{\mathfrak{p}}_v^{2\tilde{\delta}_v+2}) = q_v^{\tilde{\delta}_v}$ choices for u and, for each choice of u, $\#\ (\tilde{\mathfrak{p}}_v^{\tilde{\delta}_v+2}/\tilde{\mathfrak{p}}_v^{2\tilde{\delta}_v+2}) = q_v^{\tilde{\delta}_v}$ choices for t. This gives $q_v^{2\tilde{\delta}_v}$ solutions to (32) with $\mathrm{ord}_{\tilde{k}_v}(u) \geqslant \tilde{\delta}_v + 2$. Thus there are $2q_v^{2\tilde{\delta}_v}$ solutions in all in this case.

Finally, we must consider the case where $\tilde{\delta}_v = 2m_v + 1$. We may assume that the uniformizer, π_v, has been chosen so that $\sqrt{\pi_v} \in \tilde{k}_v$. Since $\mathrm{ord}_{\tilde{k}_v}(u) \geqslant \tilde{\delta}_v + 2$, we may write $u = (\eta_1 - \eta_2)\pi_v\bar{u}$ with $\bar{u} \in \tilde{\mathcal{O}}_v$. Again $t \equiv 1 + 2\pi_v\bar{u}\ (\tilde{\mathfrak{p}}_v^{\tilde{\delta}_v+2})$ and so $t = 1 + 2\pi_v\bar{u} + \sqrt{\pi_v}\pi_v^{m_v+1}c$ with $c \in \tilde{\mathcal{O}}_v$. Let us write $\bar{u} = \bar{u}_0 + \bar{u}_1\sqrt{\pi_v} + \bar{u}_2\pi_v$ and $c = c_0 + c_1\sqrt{\pi_v} + c_2\pi_v$ where $\bar{u}_0, \bar{u}_1, c_0, c_1 \in \mathcal{O}_v$ and $\bar{u}_2, c_2 \in \tilde{\mathcal{O}}_v$. This is possible since $\tilde{k}_v/k_v$ is ramified. A simple calculation gives

$$\begin{aligned} t - t^\sigma &= 4\bar{u}_1\sqrt{\pi_v}\pi_v + 2c_0\sqrt{\pi_v}\pi_v^{m_v+1} - 2\pi_v^2(\bar{u}_2^\sigma - \bar{u}_2) \\ &\quad + \sqrt{\pi_v}\pi_v^{m_v+2}(c_2^\sigma + c_2). \end{aligned}$$

Now $\bar{u}^\sigma - \bar{u}_2 \in \tilde{\mathfrak{p}}_v^{\tilde{\delta}_v}$ and $c_2^\sigma + c_2 = (c_2^\sigma - c_2) + 2c_2 \in \tilde{\mathfrak{p}}_v^{2m_v}$ and so the last two terms lie in $\tilde{\mathfrak{p}}_v^{2\tilde{\delta}_v+3} \subseteq \tilde{\mathfrak{p}}_v^{2\tilde{\delta}_v+2}$. Thus

$$t - t^\sigma \equiv 4\bar{u}_1\sqrt{\pi_v}\pi_v + 2c_0\sqrt{\pi_v}\pi_v^{m_v+1} \ \ (\tilde{\mathfrak{p}}_v^{2\tilde{\delta}_v+2})$$

and so $t^\sigma \equiv t \ (\tilde{\mathfrak{p}}_v^{2\tilde{\delta}_v+2})$ if and only if $\bar{u}_1 \equiv -(\pi_v^{m_v}/2)c_0 \ (\mathfrak{p}_v)$. Since u, t are determined modulo $\tilde{\mathfrak{p}}_v^{2\tilde{\delta}_v+2}$, we can regard $\bar{u}, c$ as elements of $\tilde{\mathcal{O}}_v/\tilde{\mathfrak{p}}_v^{\tilde{\delta}_v}$. There are $q_v^{2\tilde{\delta}_v-1}$ pairs $(\bar{u}, c)$ satisfying the congruences relating $\bar{u}_1$ and c_0 and these lead to $q_v^{2\tilde{\delta}_v-1}$ pairs (u, t). Thus there are $2q_v^{2\tilde{\delta}_v-1}$ solutions in all. $\square$

Proposition 5.14. *We have*

$$\mathrm{vol}(K_v w_\eta) = \tfrac{1}{2}\, q_v^{-\tilde{\delta}_v - 2\lfloor \tilde{\delta}_v/2 \rfloor}(1 - q_v^{-1})(1 - q_v^{-2})^2.$$

Proof. In light of the previous two lemmas and Proposition 5.8.6, we have

$$\begin{aligned}\mathrm{vol}(K_v w_\eta) &= q_v^{-8(\tilde{\delta}_v+1)} \cdot \frac{\# G_{\mathcal{O}_v/\mathfrak{p}_v^{\tilde{\delta}_v+1}}}{\# G_{\bar{w}_\eta\ \mathcal{O}_v/\mathfrak{p}_v^{\tilde{\delta}_v+1}}} \\ &= q_v^{-8(\tilde{\delta}_v+1)} \cdot \frac{(q_v^2 - q_v)^2 (q_v^2 - 1)^2 (q_v^{2\tilde{\delta}_v+1})^4 (q_v^{\tilde{\delta}_v})^4}{2q_v^{\tilde{\delta}_v + 2\lfloor \tilde{\delta}_v/2 \rfloor} \cdot q_v^{4\tilde{\delta}_v+3} \cdot (q_v - 1)} \\ &= \tfrac{1}{2}\, q_v^{-\tilde{\delta}_v - 2\lfloor \tilde{\delta}_v/2 \rfloor}(1 - q_v^{-1})(1 - q_v^{-2})^2. \qquad \square\end{aligned}$$

From Propositions 5.14 and 5.11.5 we easily obtain the following.

Corollary 5.15. *We have*

$$\mathrm{vol}(K_v w_\ell^{\mathrm{ur}}) = q_v^{-\tilde{\delta}_v}(1 - \tfrac{1}{2}\, q_v^{-2\lfloor \tilde{\delta}_v/2 \rfloor})(1 - q_v^{-1})(1 - q_v^{-2})^2.$$

This completes the verification of the values of $\bar{\varepsilon}_v(x)$ in Tables 1 and 2.

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